ON UDRIŞTE ODE AND A PROBLEM OF JIANG

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Într-o lucrare anterioară [1]*, am introdus o nouă clasă de funcţii autoconcordante, definite pe o varietate Riemanniană, înzestrată cu metrică de tip diagonal. Scopul acestei note este dublu. Întâi stabilim o clasă de funcţii autoconcordante, care generează o anumită metrică. Apoi, dăm răspuns unei probleme deschise de Prof. Danchi Jiang referitoare la motivaţia studiului funcţiilor autoconcordante pe varietăţi. Astfel, sunt completate rezultate recente datorate primului şi celui de-al treilea autor, publicate în Balkan J. Geom. Appl.,* 2009*.*

In our paper [1]*, we introduced a new class of self-concordant functions, defined on Riemannian manifolds endowed with metrics of diagonal type. The aim of this note is twofold. First we introduce a class of self-concordant functions which generate the given metric. Then we give positive answer to an open problem by Prof. Danchi Jiang giving a motivation to the study of self-concordant functions on manifolds. This finalizes some recent results due to the first and to the third author, published in Balkan J. Geom. Appl.,* 2009*.*

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1. Introduction and preliminaries

The notion of self-concordant function is studied on Riemannian manifolds due to the necessity to develop optimization methods. Recent works by famous scientists suggest that such kind of methods are better understood on manifolds rather on than Euclidean space. To have some examples, see [7] for Newton-type algorithms; [5], [6] for interior point methods; [15] for the Riemannian context of optimization methods.

Given (M, g) a Riemannian manifold, we denote by ∇ the Levi-Civita connection induced by the metric g.

Consider a function $f\colon M \to \mathbb{R}$, defined on an open domain, as closed mapping, that is $\{(f(P), P),\ P \in \mathrm{Dom}\,(f)\}$ is a closed set in the product manifold $\mathbb{R} \times M$. Suppose f be at least three times differentiable.

According to [7] and [13], we introduce

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Definition 1.1. The function f is said to be *k-self-concordant*, $k \geq 0$, with respect to the Levi-Civita connection ∇ defined on M if the following condition holds:

$$\left|\nabla^3 f(x)(X_x, X_x, X_x)\right| \leq 2k \left(\nabla^2 f(x)(X_x, X_x)\right)^{\frac{3}{2}}, \quad \forall x \in M,\ \forall X_x \in T_x M.$$

Remark 1.1. Given the metric g, and inspired by the linearity of the set of self-concordant functions [8], in our work [1], we used decomposable functions $f\colon \mathbb{R}^n_+ \to \mathbb{R}$, of the form

$$f(x^1, x^2, \ldots, x^n) = f_1(x^1) + f_2(x^2) + \cdots + f_n(x^n), \tag{1}$$

to find a new class of self-concordant functions. Here $f_i\colon \mathbb{R}_+ \to \mathbb{R}$ are differentiable functions.

2. Udrişte ODE

Let be given the Riemannian manifold $M = \mathbb{R}^n_+$, endowed with the diagonal metric

$$g(x^1, x^2, \ldots, x^n) = \operatorname{diag}\left(\frac{1}{g_1^2(x^1)}, \frac{1}{g_2^2(x^2)}, \cdots \frac{1}{g_n^2(x^n)}\right), \tag{2}$$

where the functions $\dfrac{1}{g_i}$ admit upper bounded primitives. These metrics of diagonal type are particular cases of Hessian type metrics, [2], [13]. Indeed, the decomposable function $H = \sum_{i=1}^{n} H_i(x^i)$, satisfies

$$\frac{\partial^2 H}{\partial x^i \partial x^j} = H_i''(x^i)\delta_{ij}, \quad i = \overline{1,n},\ j = \overline{1,n}.$$

It is interesting to remark that such kind of metrics are used by Papa Quiroz [9] and Rapcsák [10], [11] to solve wide classes of problems arising from linear optimizations and nonlinear optimizations, respectively. Moreover, Hessian type metrics are useful tools in solving specific problems of WDVV PDEs of string theory [3].

Consider the case of differential equality in Definition 1.1, and taking into account the form of function f in (1), we have

Theorem 2.1. *Let us suppose that the manifold $M = \mathbb{R}^n_+$ is endowed with the diagonal metric* (2)*, and for each $i = \overline{1,n}$ the function g_i satisfies the inequalities $\displaystyle\int \frac{1}{g_i(x^i)} dx^i < 0$ (no summation). If*

$$f_i(x^i) = \frac{1}{k^2} \int \left[\frac{1}{g_i(x^i)} \int \frac{1}{g_i(x^i)\left(\displaystyle\int \frac{1}{g_i(x^i)} dx^i\right)^2} dx^i\right] dx^i, \qquad i = \overline{1,n}. \tag{3}$$

then the decomposable function f, defined by (1)*, is k-self-concordant.*

We now change the above point of view, according to a personal communication of Prof. Dr. Constantin Udrişte when work [1] was in preparation.

We ask to find decomposable functions f as in (1) which satisfy Theorem 2.1 and generate the metric g in (2), that is we have $\frac{1}{g_i} = f_i''$, for all indices $i = \overline{1, n}$. At Professor Udrişte's suggestion, we call such kind of function *self self-concordant.*

After a technical computation, we get the criterion in

Theorem 2.2 (UDRIŞTE ODE). *Suppose the functions f_i, $i = \overline{1, n}$ are given by the formulas* (3). *Then a sufficient condition for the function f in* (1) *to be self self-concordant is that each f_i is the solution of the following* ODE

$$(f_i' + \alpha) \exp\left(k^2 f_i\right) - \beta \exp\left(\gamma f_i'\right) = 0, \qquad i = \overline{1, n} \tag{4}$$

where α, β, and γ are real constants.

Remark 2.1. Theorem 2.2 indicates a wide class of self self-concordant functions. As the reader can see, for $\alpha = \gamma = 0$ and $\beta = 1$ one obtains Theorem 2.4 in [1] which claims that the Shannon entropy [12] is a self self-concordant function.

3. **Open problem by Jiang**

In a personal communication addressed to the third author on April 29, 2009, Professor Danchi Jiang (School of Engineering, University of Tasmania, Australia) asks:

Can we find a self-concordant function on Riemannian manifold, which cannot be represented as a self-concordant function in Euclidean space?

The answer to this problem is important for the proper justification of the significance of the research of self-concordant function on manifolds.

• Let $\mathbb{R}$ be the set of all real numbers and x a point on $\mathbb{R}$. According to [17] a Riemannian metric on $\mathbb{R}$ is a function $g\colon \mathbb{R} \to (0, \infty)$, which is assumed of C^∞-class. This metric yields the linear connection $\Gamma(x) = \frac{d}{dx} \ln \sqrt{g(x)}$. Given a function f, at least two times differentiable, then the Hessian has the form $\nabla^2 f = f'' - \Gamma f'$.

CASE 1. Suppose $M = \mathbb{R}$ endowed with the metric $g(x) = \exp(2x)$. In this case $\Gamma(x) = 1$, for all real x. Consider the real function $f(x) = \exp(x)$. It follows $\nabla^2 f = 0$ and $\nabla^3 f = 0$, therefore f is self-concordant. If we consider on $M = \mathbb{R}$ the Euclidean metric, $g(x) = 1$, then $\Gamma(x) = 0$, for all real x, and the self-concordance condition for the same function f leads to $1 \le 4k^2 \exp(x)$, which is not true for all $x \in \mathbb{R}$.

CASE 2. Let us suppose that $M = (0, \infty)$ is endowed with the metric $g(x) = \frac{1}{x^2}$. In this case $\Gamma(x) = -\frac{1}{x}$, for all real $x \in (0, \infty)$. On $(0, \infty)$, consider the real function $f(x) = -\ln x$. It follows $\nabla^2 f = 0$ and $\nabla^3 f = 0$, therefore f is k-self-concordant. If we consider on $M = (0, \infty)$ the Euclidean metric, $g(x) = 1$, then the self-concordance condition for the same function f leads to $k \geq 1$. Therefore, the function f is not k-self-concordant with respect to the Euclidean metric if $k \in (0, 1)$.

Remark 3.1. The two cases above were supposed to be elementary examples. We underly that the considered functions are linear geodesic.

- Let us consider the Poincaré plane

$$H = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}, \qquad g_{ij} = \frac{1}{y^2}\delta_{ij}.$$

The components of the Riemannian connection are

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y}, \quad \Gamma^2_{11} = \frac{1}{y}.$$

If we consider a function f on (H, g_{ij}), and we denote

$$f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^s_{ij}\frac{\partial f}{\partial x^s}, \quad f_{,ijk} = f_{,ij,k} = \partial_k f_{,ij} - f_{,sj}\Gamma^s_{ki} - f_{,si}\Gamma^s_{kj}$$

then we have

$$\nabla^2 f(x)(X_x, X_x) = f_{,ij}X^i X^j, \qquad \nabla^3 f(x)(X_x, X_x, X_x) = f_{,ijk}X^i X^j X^k.$$

If $f(x, y) = \dfrac{1}{y}$ and $X_x = (u, v)$, after some calculations, we find

$$\nabla^2 f(x, y)(X_x, X_x) = \frac{u^2 + v^2}{y^3}, \quad \nabla^3 f(x, y)(X_x, X_x, X_x) = -\frac{v(u^2 + v^2)}{y^4}.$$

Directly from Definition 1.1, we get

Proposition 3.1. *On the set* $A = \{(x, y) \mid x \in \mathbb{R}, \quad 0 < y \leq 4k^2\}$, *the function* $f(x, y) = \dfrac{1}{y}$ *is* k*-self-concordant.*

- Consider the Euclidean plane

$$M = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}, \qquad g_{ij} = \delta_{ij}.$$

The connection components are $\Gamma^k_{ij} = 0$.

If $f(x, y) = \dfrac{1}{y}$ and $X_x = (u, v)$, after some calculations, we find

$$\nabla^2 f(x, y)(X_x, X_x) = \frac{2v^2}{y^3}, \quad \nabla^3 f(x, y)(X_x, X_x, X_x) = -\frac{6v^3}{y^4}.$$

Using Definition 1.1, we can see that the function f is k-self-concordant if $y \leq \dfrac{8k^2}{9}$, and we have

Proposition 3.2. *On the set* $A = \{(x,y) \,|\, x \in \mathbb{R}, \quad \frac{8k^2}{9} < y \leq 4k^2\}$, *the function* $f(x,y) = \frac{1}{y}$ *is* k*-self-concordant on the Poincaré plane but is not* k*-self-concordant on the Euclidean plane.*

The results in this section (for example, Proposition 3.1 for the Poincaré plane), give a positive answer to the problem of Danchi Jiang. Namely,

There exist self-concordant functions on Riemannian manifolds which cannot be represented as self-concordant functions in Euclidean spaces.

According to [1], [14] we have

Definition 3.1. The function f is said to be *c-self-concordant barrier*, $c \geq 0$, with respect to the Levi-Civita connection ∇ defined on M if the following condition holds:

$$(df(x)(X_x))^2 \leq c\,\nabla^2 f(x)(X_x, X_x), \quad \forall x \in M, \ \forall X_x \in T_xM.$$

After a straightforward calculation, we obtain

Proposition 3.3. *On the set* $A = \{(x,y) \,|\, x \in \mathbb{R}, \quad \frac{8k^2}{9} < y \leq 4k^2\}$, *the function* $f(x,y) = \frac{1}{y}$ *is* $64k^6$*-self-concordant barrier.*

Remark 3.2. The result in Proposition 3.3 is a deep one since finding self-concordant barriers is an actual problem of Optimization Theory. We emphasize the theories of Nesterov and Nemirovsky [8] which use this class of functions for developing linear and convex quadratic programs with convex quadratic constraints, and Udrişte's works [15], [17] which develop barrier methods for smooth convex programming on Riemannian manifolds.

Remark 3.3. To make a computer aided study of k-self-concordant functions we can perform symbolic computations for integrals. In this respect, we recommend the Maple software package [4], [16].

4. **Conclusions**

In one of our previous papers [1], we introduced and studied a new class of self-concordant functions, defined on Riemannian manifolds endowed with metrics of diagonal type. In this note, we have further developed the results published in Balkan J. Geom. Appl., 2009, by introducing a class of self-concordant functions which generate the given metric (udrişte ode). Also we give answer to an open problem by Jiang concerning the study motivation of self-concordant functions on manifolds. Our results give a relevant link between differential geometry and applied sciences.

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