ON SOME PROPERTIES OF RIGHT PURE (BI-QUASI-)HYPERIDEALS IN ORDERED SEMIHYPERRINGS

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In this paper, we present the concept of pure hyperideals of an ordered semihyperring and work with the k-extension of a bi-quasi hyperideal, which is a generalization of k-bi-quasi hyperideals of ordered semihyperrings. Moreover, we make an extension of pseudoorders, and we expand the theoretical aspects by studying in depth the quotient ordered semihyperrings. Our results include applications of the ordered semihyperrings to construct ordered semihyperrings, using properties of weak pseudoorders.

Keywords: algebraic hyperstructure; ordered semihyperring; (right) pure hyperideal; k-extension of a bi-quasi hyperideal.

MSC2020: 06F99, 16Y99, 20N20.

1. Introduction

By an *ordered semihypergroup* [21], we mean a semihypergroup $(S, \circ)$ with a (partial) order relation $\leq$ such that

$$a \leq b \Rightarrow a \circ c \leq b \circ c \text{ and } c \circ a \leq c \circ b \text{ for all } a, b, c \in S.$$

If $\emptyset \neq X, Y \subseteq S$, then

$$X \leq Y \Leftrightarrow \forall x \in X, \exists y \in Y; x \leq y.$$

We refer the reader to [21] for the definition and the related properties of ordered semihypergroups. In 2015, Davvaz et al. [13] pioneered to study pseudoorders in ordered semihypergroups. In [18], Gu and Tang initiated the study of ordered regular equivalence relations in ordered semihypergroups. Later, Tang et al. [34] considered this concept and further investigated several results on ordered semihypergroups. Afterward, Rao et al. [30] presented some properties of relative bi-(int-)Γ-hyperideals and used them to characterize regular ordered Γ-semihypergroups.

Algebraic hyperstructures were introduced and investigated by Marty [25] at the 8th Congress of the Scandinavian Mathematicians and studied extensively by many mathematicians. Several books [7, 8, 10, 12] have been written on hyperstructures theory. In general, hyperstructure theory has a wide range of applications in various fields, for example, see [14, 15, 16]. Davvaz and Musavi [11] defined cyclic codes and linear codes over hyperrings. The theory of semihyperring was first introduced by Vougiouklis [35] in 1990. In [23], Huang et al. presented the concept of $(\epsilon_\gamma, \epsilon_\gamma \vee q_\delta)$-fuzzy hyperideals in the context of semihyperrings. The principal notions of ordered semihyperring theory and some results can be found in [26, 27, 28]. Recently, Rao et al. [29] studied some properties of (left) k-bi-quasi hyperideals of ordered semihyperrings.

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Ameri and Hedayati [2, 20] studied a more restricted class of hyperideals in semihyperrings, which is called the class of k-hyperideals. In [19], ordered ideals were presented in the sense of ordered semiring theory. In 1992, Sen and Adhikari [31] studied some properties of k-ideals in semirings. The concept of k-extension of an ideal on a semiring $(R, \oplus, \odot)$ was introduced by Chaudhari et al. in [5, 6]. In [32], Sen and Adhikari studied some properties of maximal k-ideals of semirings.

In [4], Changphas and Sanborisoot studied a class of ideals in ordered semigroups which are called pure ideals. In [3], Changphas and Kummoon studied purity of ideals in ordered semigroups. In [22], Hila and Naka studied pure hyperradical in semihypergroups and proved some related results.

In 2019, Siribute and Sanborisoot [33] considered and proved some results on pure fuzzy ideals in ordered semigroups. Kazanci et al. were interested in studing fuzzy hyperideals for a given ordered semihyperrings [24]. Following [1, 17], an ideal A of a ring $(R, \oplus, \odot)$ is said to be right (left) pure ideal if for each $a \in A$, there exists $x \in A$ such that $a = a \odot x$ $(a = x \odot a)$. Previous studies on the pure ideals of algebraic structures motivated us to study the pure hyperideals of ordered semihyperrings. Theory of pure hyperideals in ordered semihyperrings is useful to explore new results associated with pure fuzzy hyperideals.

After an introduction, in Section 2 we summarize some notation and terminologies on ordered semihyperrings. In Section 3, we introduce the concept of right pure hyperideals of ordered semihyperrings and investigate some of their related results. Moreover, we introduce k-extension of a bi-quasi hyperideal in ordered semihyperrings. Finally, by using ordered regular equivalence relations on an ordered semihyperring we construct quotient ordered semihyperring.

2. Preliminaries

Let us first give the following definition. An algebraic hypersructure $(R, \oplus, \odot)$, where $\oplus$ and $\odot$ are hyperoperations on R, is called a *semihyperring* [35, 9] when the following axioms hold:

(1) $(R, \oplus)$ is a (commutative) semihypergroup;
(2) $(R, \odot)$ is a semihypergroup;
(3) The hyperoperation $\odot$ is both left and right distributive over the hyperoperation $\oplus$.

By *zero* of a semihyperring $(R, \oplus, \odot)$ we mean an element $0 \in R$ such that $x \oplus 0 = 0 \oplus x = \{x\}$ and $x \odot 0 = 0 \odot x = 0$ for all $x \in R$.

Definition 2.1. [26, 28] *Let $(R, \oplus, \odot)$ be a semihyperring and $\leq$ a (partial) order relation on R. Then $(R, \oplus, \odot, \leq)$ is an ordered semihyperring if*

(1) *for every $a \leq b \in R$, $a \oplus c \preceq b \oplus c$ and $c \oplus a \preceq c \oplus b$ for all $c \in R$;*
(2) *for every $a \leq b \in R$, $a \odot c \preceq b \odot c$ and $c \odot a \preceq c \odot b$ for all $c \in R$.*

Here, if $\emptyset \neq X, Y \subseteq R$, then

$$X \preceq Y \Leftrightarrow \forall x \in X, \exists y \in Y; x \leq y.$$

Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring. For a non-empty subset K of R we define

$$(K] := \{x \in R \mid x \leq k \text{ for some } k \in K\}.$$

For $K = \{k\}$, we write $(k]$ instead of $(\{k\}]$. If A and B are non-empty subsets of R, then we have (1) $A \subseteq (A]$ (so, $R = (R]$); (2) $((A]] = (A]$; (3) $(A] \odot (B] \subseteq (A \odot B]$; (4) If $A \subseteq B$, then $(A] \subseteq (B]$.

Definition 2.2. [27] *A non-empty subset K of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is said to be a left hyperideal (resp. right hyperideal) of R if*

(1) *$(K, \oplus)$ is a semihypergroup;*
(2) *$R \odot K \subseteq K$ (resp. $K \odot R \subseteq K$);*
(3) *If $x \in K$ and $R \ni y \leq x$, then $y \in K$, i.e., $(K] = K$.*

If K is both a left and a right hyperideal of R, then K is called a *hyperideal* of R.

Definition 2.3. [27] *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring. A left hyperideal I of R is called a left k-hyperideal of R if*

$$\forall a \in I, \forall x \in R, (a \oplus x) \cap I \neq \emptyset \Rightarrow x \in I.$$

A left hyperideal I of R is called a left k-hyperideal of type 1 of R if

$$\forall a \in I, \forall x \in R, a \oplus x \subseteq I \Rightarrow x \in I.$$

A right k-hyperideal (of type 1) is defined similarly. If a hyperideal I is both left and right k-hyperideal (of type 1), then I is known as a k-hyperideal (of type 1) of R. Clearly, every k-hyperideal is a k-hyperideal of type 1.

3. Right Pure (Bi-Quasi-)Hyperideals

In this section, we introduce various hyperideals of an ordered semihyperring, as follows.

(1) Right pure hyperideals,
(2) Complete hyperideals,
(3) k-Extension of a bi-quasi hyperideal.

Furthermore, we study some results and provide some explicit examples to support our definitions.

Definition 3.1. *A hyperideal K of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is said to be a left (resp. right) pure hyperideal if for every $a \in K$, there exists $b \in K$ such that $a \leq b \odot a$ (* resp. *$a \leq a \odot b$).*

Equivalent Definition. $a \in (K \odot a]$ (resp. $a \in (a \odot K]$).
If K is both left pure and right pure hyperideal, then K is said to be a pure hyperideal of R.

Definition 3.2. *A hyperideal K of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is said to be idempotent if $K = (K \odot K]$.*

Lemma 3.1. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring. If K is a right pure hyperideal of R, then K is idempotent.*

Proof. We have

$$K \odot K \subseteq K \odot R \subseteq K.$$

It implies that

$$(K \odot K] \subseteq (K] = K.$$

Now, let $x \in K$. Since K is a right pure hyperideal of R, there exists $y \in K$ such that $x \leq x \odot y \subseteq K \odot K$. So, $x \leq u$ for some $u \in K \odot K$. This means that $x \in (K \odot K]$. Hence, $K \subseteq (K \odot K]$. This completes the proof. □

Definition 3.3. *A hyperideal K of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is said to be complete if $K = (K \odot R] = (R \odot K]$.*

Theorem 3.1. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring. If K is a right pure hyperideal of R, then K is complete.*

Proof. Let K be a right pure hyperideal of an ordered semihyperring R. By Lemma **3**.1, K is idempotent. So, we get

$$K = (K \odot K] \subseteq (K \odot R] \subseteq (K] = K,$$

$$K = (K \odot K] \subseteq (R \odot K] \subseteq (K] = K.$$

Thus, $(K \odot R] = K = (R \odot K]$. Therefore, K is complete. □

Definition 3.4. *A hyperideal K of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is said to be $\mathfrak{J}$-pure if $K \cap (I \odot R] = (I \odot K]$ and $K \cap (R \odot I] = (K \odot I]$ for any hyperideal I of R. An ordered semihyperring R is said to be $\mathfrak{J}^*$-pure if every hyperideal of R is $\mathfrak{J}$-pure.*

In the following, we see that under some conditions (for example, in $\mathfrak{J}^*$-pure ordered semihyperrings) complete hyperideals coincide with idempotent hyperideals.

Proposition 3.1. *Let K be a hyperideal of a $\mathfrak{J}^*$-pure ordered semihyperring $(R, \oplus, \odot, \leq)$. Then, K is complete if and only if K is idempotent.*

Proof. If K is complete, then

$$K = K \cap K = K \cap (K \odot R] = (K \odot K]$$

This shows that $K = (K \odot K]$. Hence, K is idempotent.

If K is idempotent, then by the proof of Theorem **3**.1, K is complete. □

Lemma 3.2. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring. If K, I are right pure hyperideals of R, then $K \cap I$ is a right pure hyperideal of R.*

Proof. First ,we show that $K \cap I = (K \odot I]$. Since K is a right hyperideal of R, we have $K \odot I \subseteq K \odot R \subseteq K$. It follows that $(K \odot I] \subseteq (K] = K$. On the other hand, $K \odot I \subseteq R \odot I \subseteq I$. So, $(K \odot I] \subseteq (I] = I$. Hence, $(K \odot I] \subseteq K \cap I$. Now, let $a \in K \cap I$. Since I is a right pure hyperideal of R, there exists $x \in I$ such that $a \leq a \odot x \subseteq K \odot I$. So, $a \in (K \odot I]$ and hence $K \cap I \subseteq (K \odot I]$. Therefore, $(K \odot I] = K \cap I$.

Clearly, $K \cap I$ is a hyperideal of R. Let $a \in K \cap I$. Since $a \in K$ and K is a right pure hyperideal of R, there exists $b \in K$ such that $a \leq a \odot b$. Similarly, there exists $c \in I$ such that $a \leq a \odot c$. So, we have

$$a \leq a \odot c \leq (a \odot b) \odot c = a \odot (b \odot c).$$

Since $b \odot c \subseteq K \odot I \subseteq (K \odot I] = K \cap I$, we get $K \cap I$ is a right pure hyperideal of R. □

Definition 3.5. *A proper right pure hyperideal K of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is said to be purely prime if for any right pure hyperideals A_1 and A_2 of R, $A_1 \cap A_2 \subseteq K$ implies $A_1 \subseteq K$ or $A_2 \subseteq K$.*

Theorem 3.2. *Let K be a right pure hyperideal of an ordered semihyperring $(R, \oplus, \odot, \leq)$ and $x \in R$ such that $x \notin K$. Then, there exists a purely prime hyperideal P of R such that $K \subseteq P$ and $x \notin P$.*

Proof. Let $\Lambda = \{P : P$ is a right pure hyperideal of $R, K \subseteq P$ and $x \notin P\}$. Since $K \in \Lambda$, we get $\Lambda \neq \emptyset$. Also, Λ is an ordered set under the usual inclusion. Let $\{P_i : i \in \Omega\}$ be a chain in Λ. Then $P_i \subseteq P_j$ or $P_j \subseteq P_i$, for all $i, j \in \Omega$. We assert that $\bigcup\limits_{i\in\Omega} P_i$ is a right pure hyperideal of R. Clearly, $0 \in \bigcup\limits_{i\in\Omega} P_i$. Then, $\emptyset \neq \bigcup\limits_{i\in\Omega} P_i$. Let $a, b \in \bigcup\limits_{i\in\Omega} P_i$. Then $a \in P_i$ and $b \in P_j$ for some $i, j \in \Omega$. If, say, $P_i \subseteq P_j$, then both a, b are inside P_j. Since P_j is a hyperideal of R, we get $a \oplus b \subseteq P_j$. It implies that $a \oplus b \subseteq \bigcup\limits_{i\in\Omega} P_i$. If, say, $P_j \subseteq P_i$, then both a, b are inside P_i, so we have $a \oplus b \subseteq P_i$. This means that $a \oplus b \subseteq \bigcup\limits_{i\in\Omega} P_i$. Similarly, we have $a \odot b, b \odot a \subseteq \bigcup\limits_{i\in\Omega} P_i$. Also, $(\bigcup\limits_{i\in\Omega} P_i) \odot R = \bigcup\limits_{i\in\Omega} P_i \odot R \subseteq \bigcup\limits_{i\in\Omega} P_i$ and $R \odot (\bigcup\limits_{i\in\Omega} P_i) = \bigcup\limits_{i\in\Omega} R \odot P_i \subseteq \bigcup\limits_{i\in\Omega} P_i$. So, for each $a \in \bigcup\limits_{i\in\Omega} P_i$ and $r \in R$, $a \odot r \subseteq \bigcup\limits_{i\in\Omega} P_i$. Similarly, $r \odot a \subseteq \bigcup\limits_{i\in\Omega} P_i$. On the other hand,

$$(\bigcup_{i\in\Omega} P_i] = \bigcup_{i\in\Omega} (P_i] = \bigcup_{i\in\Omega} P_i.$$

Hence, $\bigcup_{i\in\Omega} P_i$ is a hyperideal of R. Now, let $a \in \bigcup_{i\in\Omega} P_i$. Then, $a \in P_i$ for some $i \in \Omega$. Since P_i is a right pure hyperideal of R, there exists $b \in P_i \subseteq \bigcup_{i\in\Omega} P_i$ such that $a \leq a \odot b$. Therefore, $\bigcup_{i\in\Omega} P_i$ is a right pure hyperideal of R. Since each $P_i \in \Lambda$ contains K and $x \notin P_i$, we have $K \subseteq \bigcup_{i\in\Omega} P_i$ and $x \notin \bigcup_{i\in\Omega} P_i$. Hence $\bigcup_{i\in\Omega} P_i \in \Lambda$ is an upper bound for chain $\{P_i : i \in \Omega\}$. By Zorn's Lemma, Λ has a maximal element, say M, such that M is a right pure hyperideal of R, $K \subseteq M$ and $x \notin M$. Now, we claim that M is a purely prime hyperideal of R. Let A_1 and A_2 be right pure hyperideals of R such that $A_1 \cap A_2 \subseteq M$, $A_1 \nsubseteq M$ and $A_2 \nsubseteq M$. Then $M \subsetneq A_i \cup M (i = 1, 2)$. By the maximality of M, we have $x \in A_1 \cup M$ and $x \in A_2 \cup M$. Since $x \in A_i \cup M$ and $x \notin M$, we have $x \in A_1 \cap A_2$. Since $x \notin M$, we get $A_1 \cap A_2 \nsubseteq M$, which is a contradiction. Therefore, M is a purely prime hyperideal of R. □

A non-empty subset A of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is called a *left (resp. right) bi-quasi hyperideal* [29] of R if (1) A is a subsemihyperring of R; (2) $(R \odot A] \cap (A \odot R \odot A] \subseteq A$ (resp. $(A \odot R] \cap (A \odot R \odot A] \subseteq A$); (3) $(A] \subseteq A$.

Remark 3.1. *A bi-quasi hyperideal A of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is called a k-bi-quasi hyperideal if for any $a \in A$ and $x \in R$, from $(a \oplus x) \cap A \neq \emptyset$ it follows $x \in A$. It is clear that every k-bi-quasi hyperideal of an ordered semihyperring R is a bi-quasi hyperideal of R.*

Next example shows that the converse of Remark **3**.1 does not hold in general.

Example 3.1. *Set $R = \{a, b, c, d\}$. Define the hyperaddition $\oplus$, multiplication $\odot$ and the (partial) order relation $\leq$ on R as follows:*

$\oplus$	a	b	c	d
a	a	b	c	d
b	b	$\{b, c\}$	c	d
c	c	c	$\{a, c\}$	d
d	d	d	d	$\{a, d\}$

$\odot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	d	d	d

$$\leq := \{(a, a), (b, b), (c, c), (d, d)\}.$$

Under the identity relation, $x \leq y$ if and only if $x = y$, it is observed that $(R, \oplus, \odot, \leq)$ is an ordered semihyperring. Put $A = \{a, d\}$. We have

(1) *A is a bi-quasi hyperideal of R. Indeed:*

$$\begin{aligned}(A \cdot R] \cap (A \cdot R \cdot A] &\subseteq ((A \cdot R) \cdot A] \\ &\subseteq (A \cdot A] \\ &\subseteq (A] \\ &= A.\end{aligned}$$

Clearly, A is not a k-bi-quasi hyperideal of R. Indeed:

$$d \oplus c = d \in A \text{ but } c \notin A.$$

(2) *A is not a left hyperideal of R, since $R \odot A = R \odot \{a, d\} = R \nsubseteq A$.*

Definition 3.6. *Let A be a left bi-quasi hyperideal of an ordered semihyperring $(R, \oplus, \odot, \leq)$. A left bi-quasi hyperideal K of R with $A \subseteq K$ is said to be a left k-extension of A, if*

$$\forall a \in A, \forall r \in R,\ (a \oplus r) \cap K \neq \emptyset \Rightarrow r \in K.$$

A right k-extension of A is defined similarly. If a hyperideal K is both left and right k-extension of A, then K is known as a k-extension of A.

Definition 3.7. *Let A be a bi-quasi hyperideal of an ordered semihyperring $(R, \oplus, \odot, \leq)$. A bi-quasi hyperideal K of R with $A \subseteq K$ is said to be a k-extension of type 1 of A, if*

$$\forall a \in A, \forall r \in R,\ (a \oplus r) \subseteq K \Rightarrow r \in K.$$

Remark 3.2. *Every k-bi-quasi hyperideal A of an ordered semihyperring R containing a bi-quasi hyperideal I of R is a k-extension of I. Also, every bi-quasi hyperideal of an ordered semihyperring R is a k-extension of $\{0\}$.*

Example 3.2. *In Example **3**.1, $K = \{a, d\}$ is a right k-extension of $I = \{a\}$.*

Definition 3.8. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring. A left bi-quasi hyperideal K of R is said to be a left pure left bi-quasi hyperideal if*

$$\forall x \in K, \exists y \in K \text{ s.t. } x \leq y \odot x \text{ i.e., } x \in (K \odot x].$$

Right pure right bi-quasi hyperideals are defined similarly.

Example 3.3. *In Example **3**.1, it is easy to show that $K = \{a, d\}$ is a pure bi-quasi hyperideal of R.*

Example 3.4. *Consider the ordered semihyperring R from Example **3**.1. Then $I = \{a, b, c\}$ is a pure right bi-quasi hyperideal of R.*

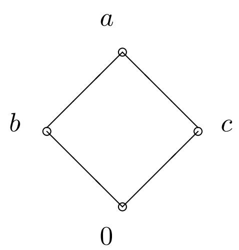
Example 3.5. *Set $R = \{0, a, b, c\}$. Define the hyperoperations $\oplus$, $\odot$ and (partial) order relation $\leq$ on R as follows:*

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	$\{0, b\}$	$\{0, b, c\}$
c	c	a	$\{0, b, c\}$	$\{0, c\}$

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	$\{0, b\}$	0
b	0	0	0	0
c	0	$\{0, c\}$	0	0

$$\begin{aligned}\leq \ := \{&(0, 0), (0, a), (0, b), (0, c), (a, a),\\ &(b, a), (b, b), (c, a), (c, c)\}.\end{aligned}$$

Then, $(R, \oplus, \odot, \leq)$ is an ordered semihyperring. The covering relation and the figure of R are given by $\prec = \{(0, b), (0, c), (b, a), (c, a)\}$.



(1) *Clearly, $B = \{0, b, c\}$ is a k-extension of a bi-quasi hyperideal $A = \{0, b\}$.*
(2) *It is easy to check that $A = \{0, b\}$ is not a k-bi-quasi hyperideal of R. Indeed:*

$$b \oplus c = \{0, b, c\} \cap A \neq \emptyset \text{ and } b \in A \text{ but } c \notin A.$$

Let $I = \{0\}$. Then, A is a k-extension of I.

(3) *It is easy to see that A is a k-bi-quasi hyperideal of type 1 of R, but it is not a k-bi-quasi hyperideal of R.*

Example 3.6. *Let us continue with the semihyperring $(R, \oplus, \odot)$ in Example **3**.5. We set*

$$\leq := \{(0, 0), (a, a), (b, b), (c, c)\}.$$

Then, $(R, \oplus, \odot, \leq)$ is an ordered semihyperring. $K = \{0, a, c\}$ is a right pure left hyperideal of R, since $0 \leq 0 \odot a$, $a \leq a \odot a$ and $c \leq c \odot a$. But $\{0, a, b\}$ is not pure right hyperideal of R, since there is no element $x \in \{0, a, b\}$ such that $b \leq b \odot x$.

Definition 3.9. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring. A relation σ on R is called a weak pseudoorder if*

(1) $\leq \subseteq \sigma$;
(2) $(a, b) \in \sigma$ *and* $(b, c) \in \sigma$ *imply* $(a, c) \in \sigma$ *for all* $a, b, c \in R$;
(3) $(a, b) \in \sigma$ *implies* $(a \oplus x)\overrightarrow{\sigma}(b \oplus x)$ *and* $(x \oplus a)\overrightarrow{\sigma}(x \oplus b)$ *for all* $a, b, x \in R$;
(4) $(a, b) \in \sigma$ *implies* $(a \odot x)\overrightarrow{\sigma}(b \odot x)$ *and* $(x \odot a)\overrightarrow{\sigma}(x \odot b)$ *for all* $a, b, x \in R$;
(5) $(a, b) \in \sigma$ *and* $(b, a) \in \sigma$ *imply* $(a \oplus x)\widetilde{\sigma}(b \oplus x)$ *and* $(x \oplus a)\widetilde{\sigma}(x \oplus b)$ *for all* $a, b, x \in R$;
(6) $(a, b) \in \sigma$ *and* $(b, a) \in \sigma$ *imply* $(a \odot x)\widetilde{\sigma}(b \odot x)$ *and* $(x \odot a)\widetilde{\sigma}(x \odot b)$ *for all* $a, b, x \in R$.

Here, if U and V are non-empty subsets of S, then

$$U\overrightarrow{\sigma}V \Leftrightarrow \forall u \in U, \exists v \in V; (u, v) \in \sigma,$$

and

$$U\widetilde{\sigma}V \Leftrightarrow \forall u \in U, \exists v \in V; (u, v) \in \sigma \text{ and } (v, u) \in \sigma \ \& \ \forall v' \in V, \exists u' \in U; (v', u') \in \sigma \text{ and } (u', v') \in \sigma.$$

A relation σ on R is said to be a *pseudoorder* if for all $a, b, c \in R$, it satisfies (1), (2), (3) $(a, b) \in \sigma$ implies $(a \oplus c)\overline{\overline{\sigma}}(b \oplus c)$ and $(c \oplus a)\overline{\overline{\sigma}}(c \oplus b)$, and (4) $(a, b) \in \sigma$ implies $(a \odot c)\overline{\overline{\sigma}}(b \odot c)$ and $(c \odot a)\overline{\overline{\sigma}}(c \odot b)$. Note that if U and V are non-empty subsets of R, then

$$U\overline{\overline{\sigma}}V \Leftrightarrow \forall u \in U, \forall v \in V; (u, v) \in \sigma.$$

The main purpose of the following theorem is to present a strategy to construct of ordered semihyperrings based on the ordered regular equivalence relations.

Theorem 3.3. *Let σ be a weak pseudoorder on an ordered semihyperring $(R, +, \cdot, \leq)$. Then, there exists a regular equivalence relation*

$$\sigma^* = \{(a, b) \in R \times R \mid a\sigma b \text{ and } b\sigma a\}$$

on R such that $(R/\sigma^*, \oplus, \odot, \preceq)$ is an ordered semihyperring, where

$$\preceq := \{(\sigma^*(x), \sigma^*(y)) \in R/\sigma^* \times R/\sigma^* \mid \exists a \in \sigma^*(x), \exists b \in \sigma^*(y) \text{ such that } (a, b) \in \sigma\}.$$

Proof. The proof is similar to the proof of Theorem 3.8 in [34]. □

In the following, we construct an ordered regular equivalence relation σ^* on an ordered semihyperring $(R, \oplus, \odot, \leq)$ such that the corresponding quotient structure $(R/\sigma^*, \oplus_{\sigma^*}, \odot_{\sigma^*}, \preceq_{\sigma^*})$ is also an ordered semihyperring. Throughout this example the notation R stands for an ordered semihyperring without zero.

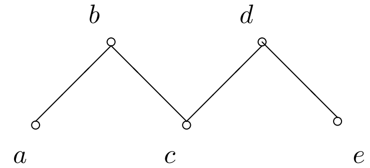
Example 3.7. *Let $R = \{a, b, c, d, e\}$. Now, to make this set more interesting, we want to define an ordered semihyperring. Define the hyperoperations $\oplus, \odot$ and (partial) order relation $\leq$ on R as follows:*

$\oplus$	a	b	c	d	e
a	$\{b, c\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	e
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	e
c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	e
d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	e
e	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	e

$\odot$	a	b	c	d	e
a	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
e	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$

$$\leq \ := \{(a, a), (a, b), (b, b), (c, b), (c, c), (c, d), (d, d), (e, d), (e, e)\}.$$

Then, $(R, \oplus, \odot, \leq)$ *is an ordered semihyperring. The covering relation and the figure of* R *are given by* $\prec = \{(a,b),(c,b),(c,d),(e,d)\}$.



We observe that R *has no proper* 2*-hyperideal. Set*

$$\begin{aligned}\sigma \ &:= \{(a,a),(a,b),(b,a),(b,b),(c,b),(c,c),(c,d),(d,c),\\ &(d,d),(e,d),(e,e)\}.\end{aligned}$$

Then σ *is a weak pseudoorder on* R*. Clearly,* σ *is not a pseudoorder, since* $(a \oplus a)\overline{\overline{\sigma}}(b \oplus a)$ *doesn't hold. Indeed:*

$$(a,b) \in \sigma, a \oplus a = \{b,c\} \text{ and } b \oplus a = \{b,d\} \text{ but } (b,d) \notin \sigma.$$

We can now set

$$\begin{aligned}\sigma^* \ &= \{(a,b) \in R \times R \mid (a,b) \in \sigma \text{ and } (b,a) \in \sigma\}\\ &= \{(a,a),(a,b),(b,a),(b,b),(c,c),\\ &= \{(c,d),(d,c),(d,d),(e,e)\}.\end{aligned}$$

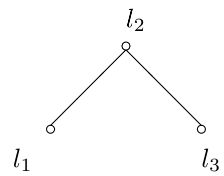
If we take $l_1 = \{a,b\}$*,* $l_2 = \{c,d\}$*, and* $l_3 = \{e\}$*, then* $R/\sigma^* = \{l_1,l_2,l_3\}$ *is still an ordered semihyperring by Theorem* **3**.*3, where* $\oplus_{\sigma^*}$*,* $\odot_{\sigma^*}$ *and* $\preceq_{\sigma^*}$ *are defined by:*

$\oplus_{\sigma^*}$	l_1	l_2	l_3
l_1	$\{l_1,l_2\}$	$\{l_1,l_2\}$	l_3
l_2	$\{l_1,l_2\}$	$\{l_1,l_2\}$	l_3
l_3	$\{l_1,l_2\}$	$\{l_1,l_2\}$	l_3

$\odot_{\sigma^*}$	l_1	l_2	l_3
l_1	$\{l_1,l_2\}$	$\{l_1,l_2\}$	$\{l_1,l_2\}$
l_2	$\{l_1,l_2\}$	$\{l_1,l_2\}$	$\{l_1,l_2\}$
l_3	$\{l_1,l_2\}$	$\{l_1,l_2\}$	$\{l_1,l_2\}$

$$\preceq_{\sigma^*} = \{(l_1,l_1),(l_1,l_2),(l_2,l_2),(l_3,l_2),(l_3,l_3)\}.$$

The covering relation and the figure of R/σ^* *are given by* $\prec = \{(l_1,l_2),(l_3,l_2)\}$.



4. Conclusions

In this paper, we studied the concept of a pure hyperideal of an ordered semihyperring. Also, we prove that if K is a hyperideal of a $\mathcal{J}^*$-pure ordered semihyperring, then K is complete if and only if K is idempotent. Also, we introduced the notion of weak pseudoorder and then we obtained some related basic results. For future work, it will be interesting to study pure fuzzy hyperideals and purely fuzzy prime hyperideals in ordered semihyperrings.

Funding

This work was supported by the National Key R&D Program of China (No. 2018YFB1005100).

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