THE STABILIZING EFFECT OF CONFINEMENT ON A LIQUID JET IN A VISCOUS OUTER FLUID

Claudiu Patrascu[1], Corneliu Balan[2]

This paper reports the effect of confinement on the instability of a liquid thread having as surrounding medium an immiscible Newtonian fluid. The analysis is made on a previously derived model [17] *which investigates the effect of an outer liquid that extends at infinity. The current approach considers, in extension, a solid cylindrical wall that bounds the external medium. We show that theoretical predictions lead to lower values for the temporal growth rate of disturbance when the rigid wall is closer to the thread. This implies that the fluid thread can be stabilized by confinement, in the limit of full confinement it being completely stable, i.e. zero growth rate for all wave numbers of disturbances. Furthermore, when the wall is located at a distance which is greater than* 10 *unperturbed jet-radii, the theoretical predictions for the growth rate are the same as for the case in which the external viscous medium extends to infinity.*

Keywords: dispersion relation, confinement, interfacial instability, growth rate

1. Introduction

Theoretical studies concerning capillary instability started with the early work of Rayleigh [15], which highlighted the role of the fastest growing mode in jet/thread break-up. This was done for an inviscid liquid thread in a quiescent external medium. It is well known that an external, immiscible, viscous fluid has a stabilizing effect [12, 17]. In terms of confinement, prior studies, both theoretical and/or experimental, have been mainly concerned with determining the role of confinement in dripping to jetting transitions [5, 6, 8]. It is assumed that the fluid thread is naturally dominated by convective instability. This claim is based on experimental evidence that confirms the existence of a "pure" convective type of instability even in the case of a confined co-flowing system [2, 3]. To what degree the fastest growing mode is affected by confinement, we wish to highlight in this study. Previous experimental studies have shown that the selection of the fastest growing mode is clearly dependent on confinement [9, 16]. Non-linear theories predict the same general trend, i.e. the break-up of the thread is delayed when confined [18]. Another stabilizing factor is the presence of elasticity in the external medium [14], a case which is not considered here.

It is worthwhile to note that confinement does not necessarily imply a stabilizing effect in general. For example, in the case of a confined Kelvin-Helmholtz flow in a plane channel, confinement has a strong destabilizing effect [1]. Furthermore, confinement can favour the transition from convective to absolute instability [10]. Somewhat the same behavior is encountered for two coaxial inviscid incompressible flows confined within a pipe, having different plug velocities [11].

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2. **Derivation of the dispersion relation**

The present paper is concerned with the effect of confinement on the dispersion relation derived by Tomotika [17]. The jet flow which we consider is schematically represented in Fig.1. The analysis starts by considering the superposition of a basic uniform flow and a perturbation $\mathbf{v} = \mathbf{v}_b + \mathbf{v}'$. We assume no velocity difference between the two media concerning the basic flow. A reference system is chosen having the velocity of the basic flow. The external medium is confined by a cylindrical wall which is suddenly brought from rest to the velocity of the basic flow, thus stationary in the moving reference system. This is done in order to avoid large velocity gradients and capture only the effects of confinement relative to the instability of the thread. A word of caution must be added, since velocity gradients do occur because the fluids are viscous, but due to the small characteristic length scale of the jet, the approximation of uniform flow still captures the relevant dynamics of the instability. The equation of motion, valid for both fluids, is represented by the Navier-Stokes equation:

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = \rho\boldsymbol{g} - \nabla p + \eta\nabla^2\mathbf{v}, \tag{1}$$

with ρ and η denoting density and viscosity, $\mathbf{v}$ and p the velocity and pressure fields. Both the inner and outer media are considered immiscible incompressible Newtonian fluids obeying $\mathrm{div}\mathbf{v} = 0$. We make use of cylindrical coordinates by considering the symmetry around the axis of the thread, therefore the projections of Eq. (1) onto r and z directions read as follows

$$\begin{aligned}
\rho\frac{\partial v_r}{\partial t} + \frac{\partial p}{\partial r} - \rho g_r &= \eta\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right),\\
\rho\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z} - \rho g_z &= \eta\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right),
\end{aligned} \tag{2}$$

where in the limit of a small perturbation of the flow field $|\mathbf{v}'| \ll |\mathbf{v}_b|$ we can neglect terms involving products of velocity components. One can eliminate the pressure field and gravity by partially differentiating the first equation with respect to z and the second one with respect to r. In this manner we are left with a single equation having as primary unknowns the velocity components.

$$\begin{aligned}
\frac{\rho}{\eta}\frac{\partial}{\partial t}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = \frac{\partial^3 v_r}{\partial z\partial r^2} + \frac{1}{r}\frac{\partial^2 v_r}{\partial z\partial r} - \frac{1}{r^2}\frac{\partial v_r}{\partial z\partial r} + \frac{\partial^3 v_r}{\partial z^3} -\\
- \frac{\partial^3 v_z}{\partial r^3} - \frac{1}{r}\frac{\partial^2 v_z}{\partial r^2} - \frac{\partial^3 v_z}{\partial r\partial z^2}
\end{aligned} \tag{3}$$

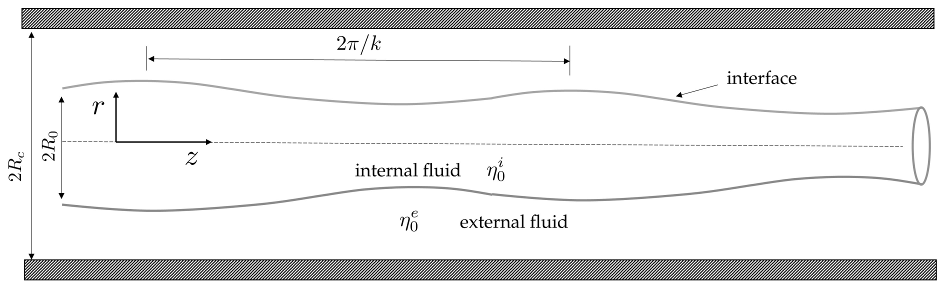

FIGURE 1. Schematic representation of a confined fluid thread surrounded by a another immiscible fluid.

Notice that, since the characteristic length scales are small (millimeters or below) the gravitational field is considered constant, therefore the derivatives of g_r and g_z vanish. Introducing the stream function, which is related to the velocity components by

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{4}$$

one can reduce the number of unknowns to one. It is assumed that the stream function has the form $\psi(r,z,t) = \Psi(r)\exp(ikz + \omega t)$, where k is the wave-number of disturbance and ω is the growth rate. The assumption stems from the hypothesis that the radius of the jet evolves as $R(z,t) = R_0 + \epsilon_0 \exp(ikz + \omega t)$, where $\epsilon_0 \ll R_0$ is a small initial perturbation of the flow.

By the introduction of the stream function the partial differential equation, obtained by coupling the two projections, now becomes an ordinary differential equation in ψ

$$\Psi^{\mathrm{IV}} - \frac{2}{r}\Psi^{\mathrm{III}} + f(k,l,r)\Psi^{\mathrm{II}} - \frac{f(k,l,r)}{r}\Psi^{\mathrm{I}} + (kl)^2\Psi = 0, \tag{5}$$

where $f(k,l,r) = 3/r^2 - k^2 - l^2$, and $l^2 = k^2 + \rho\omega/\eta$ is the modified wave-number. The general solution of Eq. (5) in terms of the r component of the stream function is [17]

$$\Psi(r) = r\left[C_1 I_1(kr) + C_2 I_1(lr) + C_3 K_1(kr) + C_4 K_1(lr)\right], \tag{6}$$

where I_1 and K_1 are the modified Bessel functions of the first and second kind, respectively, and $C_1 - C_6$ are real constants. The solution can be applied to both inner and external fluid, with the observation that for the internal fluid the general solution needs to be adapted to physically represent the phenomena in question. Since K_1 tends to infinity as r approaches 0, the stream function must not contain terms depending on K_1. The separate solutions for the two media read as follows

$$\begin{aligned}\Psi_i(r) &= r\left[C_1 I_1(kr) + C_2 I_1(l_i r)\right],\\ \Psi_e(r) &= r\left[C_3 K_1(kr) + C_4 K_1(l_e r) + C_5 I_1(kr) + C_6 I_1(l_e r)\right],\end{aligned} \tag{7}$$

with C_n, $n = \overline{1,6}$ as integration constants, indices i and e denoting internal and external fluids, respectively.

We are seeking to obtain a relation between the growth rate ω and the wave-number k, called a dispersion relation, that incorporates the effect of confinement. In order to obtain the latter, the following boundary conditions are assumed:

- no slip condition at the interface $r \approx R_0$:

$$v_z{}^e = v_z{}^i, \quad v_r{}^e = v_r{}^i, \tag{8}$$

- no slip condition at the solid wall, $r = R_c$, with R_c the radius of the cylindrical confinement geometry:

$$v_z{}^e = v_r{}^e = 0, \tag{9}$$

- continuity of tangential stresses at the interface, $r \approx R_0$:

$$\boldsymbol{n}\cdot(\boldsymbol{T}_i - \boldsymbol{T}_e)\,\boldsymbol{\tau} = 0, \tag{10}$$

- the normal stress balance at the interface, $r \approx R_0$:

$$\boldsymbol{n}\cdot(\boldsymbol{T}_e - \boldsymbol{T}_i)\,\boldsymbol{n} = \sigma\nabla\cdot\boldsymbol{n}. \tag{11}$$

where σ is the constant interfacial tension between the two immiscible fluidsand $\boldsymbol{T}_{i,e}$ represent the stress tensors of the inner and outer fluid, $\boldsymbol{n}$ and $\boldsymbol{\tau}$ being the unit normal and unit tangent vectors at the interface. Together they form a homogeneous system of six linear equations having six unknowns, $C_1 - C_6$. It is worth mentioning that confinement is introduced by Eqs. (9), where we considered that the solid wall is moving with the velocity of the base flow, thus the condition of zero velocity at the wall.

In determining the dispersion relation we proceed in the same manner as for the unbounded case of Tomotika, with which a comparison is pursued. Since the details can be found in his paper [17], we only provide a summarized version and a minor observation related to an omitted pressure term.

Making use of the stream function one can introduce the velocity components as described by Eqs. (4). Introducing these expressions in Eqs. (8) and (9) yields the first four equations needed to obtain the dispersion relation

$$\frac{\partial\psi_e}{\partial r}=\frac{\partial\psi_i}{\partial r},\ \ \frac{\partial\psi_e}{\partial z}=\frac{\partial\psi_i}{\partial z},\ \ \frac{\partial\psi_e}{\partial z}=0,\ \ \frac{\partial\psi_e}{\partial r}=0,\ \text{ at } r\approx R_0, \tag{12}$$

where $\psi_{i,e}=\Psi_{i,e}(r)\exp(ikz+\omega t)$. In a similar manner, the continuity of tangential stresses, represented by Eq. (10), can be rewritten as

$$\eta_i\left(\frac{\partial {v_r}^i}{\partial z}+\frac{\partial {v_z}^i}{\partial r}\right)=\eta_e\left(\frac{\partial {v_r}^e}{\partial z}+\frac{\partial {v_z}^e}{\partial r}\right). \tag{13}$$

In equation (11) the divergence of the normal can be computed by considering the interface as a material surface described by the equation $F=r-R(t,z)$, with $R(t,z)=R_0+\epsilon_0\exp(ikz+\omega t)$. The normal is therefore $\mathbf{n}=\nabla F/|\nabla F|$. The divergence of the normal is approximately equal to $1/R_0-\epsilon/R_0^2+\epsilon k^2$, with $\epsilon=\epsilon_0\exp(ikz+\omega t)$ being the displacement of a particle residing at the interface, which obeys $\partial\epsilon/\partial t=v_r^i$. In deriving the divergence of the normal an approximation of order ϵ_0 has been taken since $\epsilon_0\ll R_0$. Equation (11) thus becomes

$$\Delta p+2\eta_e\frac{\partial {v_r}^e}{\partial r}-2\eta_i\frac{\partial {v_r}^i}{\partial r}=\sigma\left(\frac{1}{R_0}-\frac{\epsilon}{R_0^2}+\epsilon k^2\right). \tag{14}$$

Since in the basic state, that of unperturbed flow, the pressure difference at the interface is $\Delta p=\sigma/R_0$, the first term on the right hand side cancels. In the analysis of Tomotika this pressure difference is omitted, but it can be showed that this term naturally cancels when relating it to the base flow. Incorporating the last result we obtain a system of six equations having as primary unknowns the integration constants C_n from Eqs. (7). The explicit forms of the above mentioned equations are given below and were obtained by using the relation between the velocity components and the stream function (4) and the solution provided by (7)

$$\begin{gathered}C_1kI_0(kR_0)+C_2l_iI_0(l_iR_0)+C_3kK_0(kR_0)+\\+C_4l_eK_0(l_eR_0)-C_5kI_0(kR_0)-C_6I_0(l_eR_0)=0\end{gathered} \tag{15}$$

$$\begin{gathered}C_1I_1(kR_0)+C_2I_1(l_iR_0)-C_3K_1(kR_0)-\\-C_4K_1(l_eR_0)-C_5I_1(kR_0)-C_6I_1(l_eR_0)=0,\end{gathered} \tag{16}$$

$$C_3kK_0(kR_c)+C_4l_eK_0(l_eR_c)-C_5kI_0(kR_c)-C_6I_0(l_eR_c)=0, \tag{17}$$

$$C_3K_1(kR_c)+C_4K_1(l_eR_c)+C_5I_1(kR_c)+C_6I_1(l_eR_c)=0, \tag{18}$$

$$\begin{gathered}2k^2\left[\beta C_1I_1(kR_0)-C_3K_1(kR_0)-C_5I_1(kR_0)\right]+\\+\left(k^2+{l_i}^2\right)\left[\beta C_2I_1(l_iR_0)-C_4l_eK_1(l_eR_0)-C_6I_1(l_eR_0)\right]=0,\end{gathered} \tag{19}$$

$$X_1+X_2-X_3-\frac{\omega\rho_e}{ikR_0}X_4=X_5+X_6-\frac{\omega\rho_i}{ikR_0}X_7+X_8, \tag{20}$$

where:

$$X_1=\eta_e\frac{{l_e}^2-k^2}{ikR_0}l_eR_0\left[C_6I_0(l_eR_0)-C_4K_0(l_eR_0)\right],$$

$$X_2=2i\eta_ek\left[C_5kI_0(kR_0)-\frac{C_5}{R_0}I_1(kR_0)+C_6l_eI_0(l_eR_0)-\frac{C_6}{R_0}I_1(l_eR_0)\right],$$

$$X_3 = 2i\eta_e k\left[C_3kI_0(kR_0) + \frac{C_3}{R_0}I_1(kR_0) + C_4l_eI_0(l_eR_0) + \frac{C_4}{R_0}I_1(l_eR_0)\right],$$
$$X_4 = C_6l_eR_0I_0(l_eR_0) - C_4l_eR_0K_0(l_eR_0) + C_5kR_0I_0(kR_0) - C_4kR_0K_0(kR_0),$$
$$X_5 = \eta_i\frac{{l_i}^2 - k^2}{ik}C_2l_iI_0(l_iR_0),$$
$$X_6 = 2i\eta_i k\left[C_1kI_0(kR_0) - \frac{C_1}{R_0}I_1(kR_0) + C_2l_iI_0(l_iR_0) - \frac{C_2}{R_0}I_1(l_iR_0)\right],$$
$$X_7 = C_1kR_0I_0(kR_0) + C_2l_iR_0I_0(l_iR_0),$$
$$X_8 = \frac{i\sigma}{\omega {R_0}^2}\left[(kR_0)^2 - 1\right]\left[C_1I_1(kR_0) + C_2I_1(l_iR_0)\right].$$

The system is homogeneous and we wish to obtain non-trivial solutions, thus the determinant of the coefficients must be equal to zero. In order to compare the effects of confinement to those obtained by the unbounded case we proceed by expanding all terms dependent on density in ascending powers of either ρ_i or ρ_e, considered very small but not zero. For example, the modified Bessel function of the first kind $I_0(l_iR_0)$ can be written as

$$I_0(l_iR_0) \approx I_0(kR_0) + \frac{1}{2}\frac{\rho_i\omega}{2\eta_i k}I_0'(kR_0) + \mathcal{O}(\rho_i^2), \tag{21}$$

where the same expansion has been inferred for the modified wave number $l_iR_0 \approx kR_0(1 + \rho_i\omega/2\eta_i k)$. Taking the same inertia-less limit of the model as Tomotika [17] and considering the addition of the confinement hypotheses, the dispersion relation will become a six-by-six determinantal equation. The determinantal equation is therefore

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & \cdots & a_{26} \\ \vdots & \vdots & \ddots & \vdots \\ a_{61} & a_{22} & \cdots & a_{66} \end{vmatrix} = 0. \tag{22}$$

The explicit form of a_{ij} terms in the above determinant can be found in Appendix A.

An explicit expression for the growth rate can be obtained by expanding the determinant from Eq. (22) with respect to the first row. Finally, the dispersion relation is obtained as

$$\tilde{\omega} = \frac{1}{2}(1 - \tilde{k}^2)\, f(\alpha, \beta, \tilde{k}), \tag{23}$$

where $\tilde{\omega} = \omega\eta_e R_0/\sigma$ is the non-dimensional growth rate, $\tilde{k} = kR_0$ is the non-dimensional wave-number, $\alpha = R_c/R_0$ is the confinement ratio and $\beta = \eta_i/\eta_e$ the viscosity ratio. The function $f(\alpha, \beta, \tilde{k})$ is given by:

$$f = \frac{\mathcal{D}_2\tilde{k}I_0(\tilde{k}) - (\mathcal{D}_1 + \mathcal{D}_2)\, I_1(\tilde{k})}{a_{46}\mathcal{D}_6 - a_{45}\mathcal{D}_5 + a_{44}\mathcal{D}_4 - a_{43}\mathcal{D}_3 - \mathcal{D}_1\mathcal{B}_1 + \mathcal{D}_2\mathcal{B}_2}, \tag{24}$$

where $\mathcal{D}_n = \det(a_{ij})$, with $n = \overline{1,6}$, $i = \overline{2,6}$, $j = \overline{1,6} \setminus \{n\}$, being the six determinants that result from the expansion with respect to the first row of the determinant present in Eq. (22), $\mathcal{B}_1 = \beta[\tilde{k}I_0(\tilde{k}) - I_1(\tilde{k})]$ and $\mathcal{B}_2 = \beta[\tilde{k}^2I_1(\tilde{k}) - \tilde{k}I_0(\tilde{k}) + I_1(\tilde{k})]$.

3. **Discussion**

Having determined the dispersion relation (23), we now seek to find the manner in which the confinement ratio α alters the instability of the thread. Fig.2 shows the modifications brought by increasing the value of α, on the dispersion curve. Two main observations can be made: *i)* the maximum of the dispersion curve increases as α increases, bigger values

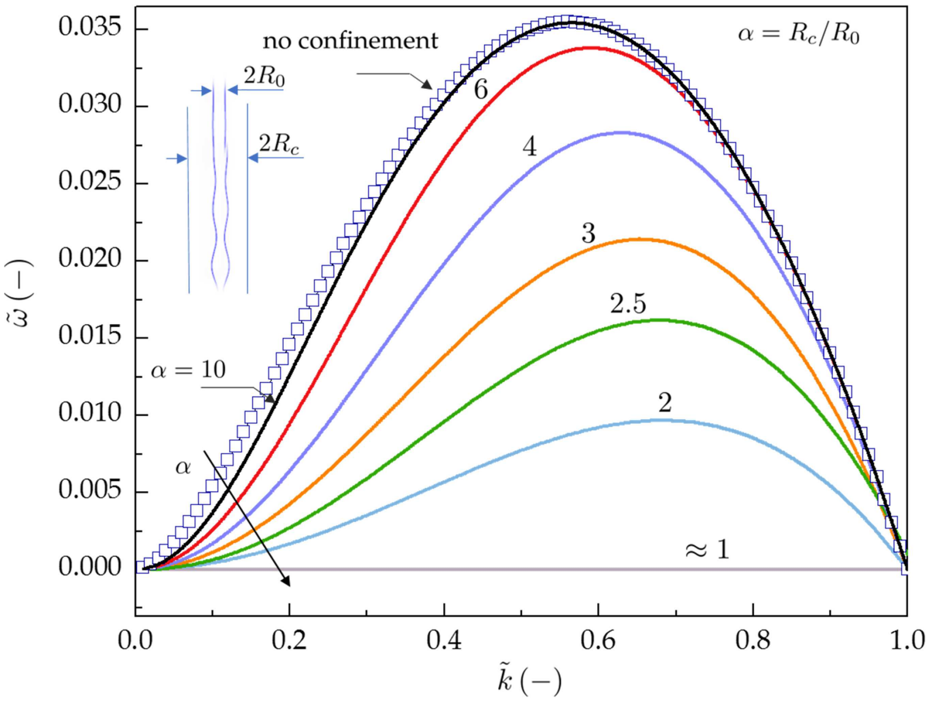

FIGURE 2. Dispersion curves relating the growth rate $\tilde{\omega}$ and the wave number $\tilde{k}$ for different values of the confinement ratio α and $\beta = 1$. Increasing the radius of confinement results in an increase of $\tilde{\omega}$, the upper limit (of no confinement) being achieved when the radius of confinement is more than 10 times greater than the radius of the unperturbed thread R_0.

in terms of growth rate implying that the instability evolves more rapidly as the thread is less confined; *ii)* the wave-number corresponding to the maximum growth rate decreases, thus longer wave-lengths are seen dominating the thread as the solid wall is further away. As the confinement ratio approaches unity, i.e. $R_c \to R_0$, the growth rate tends to zero, thus implying a "totally" stable thread. On the other hand, above $\alpha = 10$, the dispersion curves show no alteration as the confinement ratio increases, the predictions being, in this case, identical to those provided by the model of Tomotika, which is derived for an external medium extending at infinity. This suggests that the fluid thread can be considered as unconfined if the solid wall is at a distance 10 times larger than the unperturbed thread radius. The result is similar to previous theoretical investigations regarding temporal stability analysis of a viscoelastic thread under zero pressure gradient [7], with the observation that the dispersion relation is obtained by means of a steady-state Poiseuille flow.

The maximum value of the growth rate, over a wide range of viscosity ratios, decreases as the viscosity ratio increases as shown in Fig.3 . The same behaviour is encountered for any value of the confinement ratio, with the observation that lower values of the maximum growth rate are predicted as α decreases. The physical significance of the maximum value of the growth rate is related to the fastest growing mode of perturbation. The mode that has the greatest value in terms of the growth rate will become dominant, and cause thread break-up. This is, of course, an implication of linear stability theory, the final stages of break-up being dominated by non-linear effects [4].

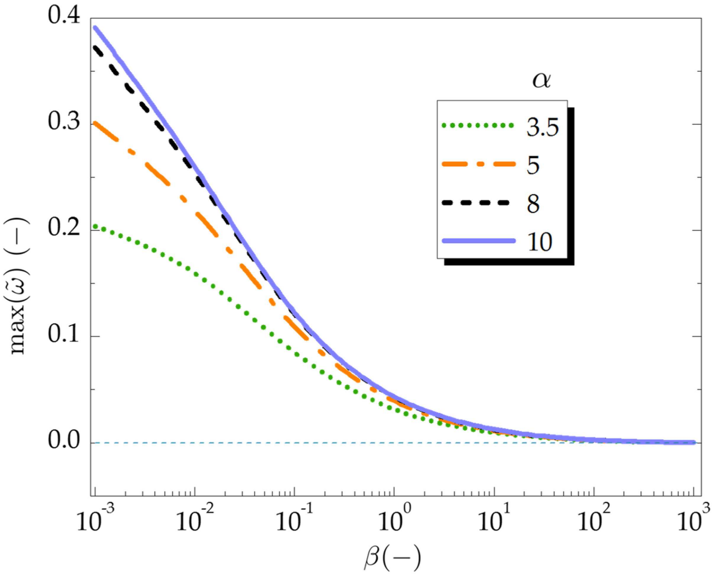

FIGURE 3. Curves of the maximum value of the growth rate as a function of both the viscosity and the confinement ratio. The graph shows decreasing values of the growth rate with a decreasing confinement ratio and an increasing viscosity ratio.

The maximum value of the growth rate is related to a single wave-number of disturbance ($\tilde{k}_M$). Fig.4 depicts the effects of the confinement ratio α, over a wide range of

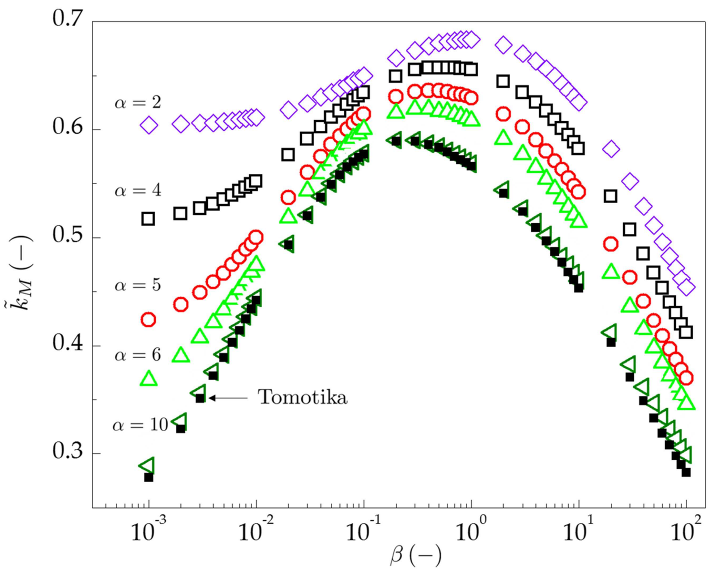

FIGURE 4. The dependence of the critical non-dimensional wave-number $\tilde{k}_M$ on the viscosity ratio β for a several values of the confinement ratio α. When $\alpha > 10$ the theoretical predictions tend to those provided by Tomotika [17] (■).

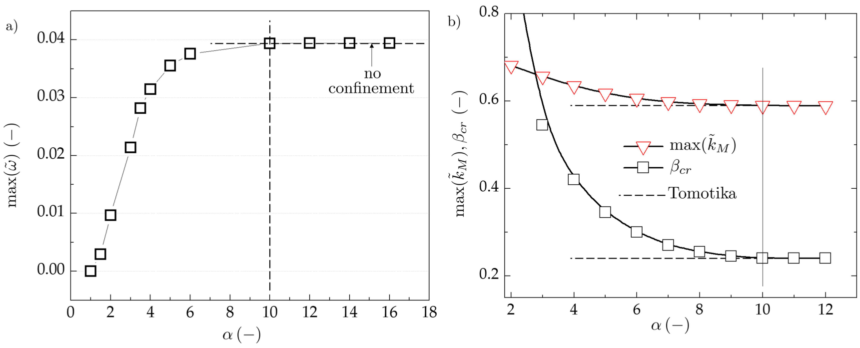

FIGURE 5. a) Curves of max($\tilde{\omega}$) as a function of the confinement ratio α. The graph shows higher values of $\tilde{\omega}$ for increasing values of α. As the latter increases above 10 it brings no change to the theoretical prediction; b) Modifications brought by the increase of α on the maximum value of the critical wave-number $\tilde{k}_M$ and the correspondent critical viscosity ratio β_{cr}.

viscosity ratios β. Increasing α results in a decrease of the critical wave-number $\tilde{k}_M$, over the entire range of viscosity ratios. When $\alpha > 10$ the prediction is identical to that of the model having no confinement. For the case where no solid wall is present, the maximum value of the critical wave-number, i.e. the minimum critical wave-length, is achieved for a critical viscosity ratio of $\beta_{cr} \approx 0.28$. When the solid wall is present, the value of the viscosity ratio that triggers this maximum value of $\tilde{k}_M$ increases. For example, for $\alpha = 2$ this viscosity ratio is close to unity. The results are similar to those obtained in [13], except that in the latter the rigid wall is stationary with respect to the liquid thread, thus making this current approach applicable to dynamically-confined liquid jets.

4. **Conclusions**

In order to summarize the present findings, Fig.5 presents the effect of the confinement ratio α on the maximum value of the growth rate $\tilde{\omega}$, the maximum value of the critical wave-number $\tilde{k}_M$ and the correspondent critical viscosity ratio β_{cr} that triggers it. The increase of the confinement ratio increases the value of the maximum growth rate, implying a faster development of the instability of the thread (Fig.5-a). Above the value of 10 the prediction is insensible to the presence of the solid wall. The same behaviour is predicted in terms of the maximum value of the critical wave-number, as depicted in Fig. 5-b. From a theoretical point of view, the fluid thread acts as if it is not subject to confinement, in terms of both growth rate and wave-number of disturbance, when the solid wall is located at a distance 10 times larger than the unperturbed thread radius.

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Appendix A. Explicit form of a_{ij} terms found in Eqn. (22)

$$
\begin{aligned}
a_{11} &= \beta\left[\tilde{k}I_0(\tilde{k}) - I_1(\tilde{k})\right] + \frac{1}{2\tilde{\omega}}\left(\tilde{k}^2 - 1\right)I_1(\tilde{k}),\\
a_{12} &= \beta\left[\tilde{k}^2 I_1(\tilde{k}) - \tilde{k}I_0(\tilde{k}) + I_1(\tilde{k})\right] + \frac{1}{2\tilde{\omega}}\left(\tilde{k}^2 - 1\right)\left[\tilde{k}I_0(\tilde{k}) - I_1(\tilde{k})\right],\\
a_{13} &= -\left[\tilde{k}K_0(\tilde{k}) + K_1(\tilde{k})\right],
\end{aligned}
$$

$$
\begin{aligned}
a_{14} &= -\left[\tilde{k}^2 K_1(\tilde{k}) - \tilde{k}K_0(\tilde{k}) + K_1(\tilde{k})\right],\\
a_{15} &= I_1(\tilde{k}) - \tilde{k}I_0(\tilde{k}),\ a_{16} = \tilde{k}I_0(\tilde{k}) - \tilde{k}^2 I_1(\tilde{k}) - I_1(\tilde{k}),\\
a_{21} &= I_0(\tilde{k}),\ a_{22} = I_0(\tilde{k}) + \tilde{k}I_1(\tilde{k}),\ a_{23} = K_0(\tilde{k}),\\
a_{24} &= K_0(\tilde{k}) - \tilde{k}K_1(\tilde{k}),\ a_{25} = -I_0(\tilde{k}),\\
a_{26} &= -I_0(\tilde{k}) - \tilde{k}I_1(\tilde{k}),\ a_{31} = \beta I_1(\tilde{k}),\ a_{32} = \beta\tilde{k}I_0(\tilde{k})\\
a_{33} &= -K_1(\tilde{k}),\ a_{34} = \tilde{k}K_0(\tilde{k}),\ a_{35} = -I_1(\tilde{k}),\ a_{36} = -\tilde{k}I_0(\tilde{k}),\\
a_{41} &= I_1(\tilde{k}),\ a_{42} = \tilde{k}I_0(\tilde{k}) - I_1(\tilde{k}),\\
a_{43} &= -K_1(\tilde{k}),\ a_{44} = K_1(\tilde{k}) + \tilde{k}K_0(\tilde{k}),\ a_{45} = -I_1(\tilde{k})\\
a_{46} &= I_1(\tilde{k}) - \tilde{k}I_0(\tilde{k}), a_{51} = 0,\ a_{52} = 0,\ a_{53} = K_1(\alpha\tilde{k}),\\
a_{54} &= -K_1(\alpha\tilde{k}) - \alpha\tilde{k}K_0(\alpha\tilde{k}),\ a_{55} = I_1(\alpha\tilde{k}),\\
a_{56} &= \alpha\tilde{k}I_0(\alpha\tilde{k}) - I_1(\alpha\tilde{k}),\ a_{61} = 0,\ a_{62} = 0,\ a_{63} = K_0(\alpha\tilde{k}),\\
a_{64} &= K_0(\alpha\tilde{k}) - \alpha\tilde{k}K_1(\alpha\tilde{k}),\ a_{65} = -I_0(\alpha\tilde{k}),\\
a_{66} &= -I_0(\alpha\tilde{k}) - \tilde{k}_c I_1(\alpha\tilde{k}).
\end{aligned}
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