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SOLVING SOME FUNCTIONAL AND OPERATORIAL EQUATIONS BY A GENERAL CONSTRUCTIVE METHOD

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În Secţiunea 2 a prezentei lucrări enunţăm versiuni îmbunătăţite ale unor rezultate publicate pentru prima dată în [30], demonstrând ceea ce este mai important din partea adăugată. Mai precis, arătăm că analiticitatea funcţiei g în jurul punctului său de minim este o condiţie suficientă pentru derivabilitatea soluţiei f în acelaşi punct (vezi Teorema 2.1). Tot în Secţiunea 2, reamintim teorema abstractă corespunzătoare de rezolvare a unor ecuaţii operatoriale, de acelaşi tip cu cele funcţionale soluţionate în Teorema 2.1 (Teorema 2.2). În Secţiunea 3, aplicăm rezultatele generale din Secţiunea 2 la rezolvarea unei ecuaţii funcţionale concrete (Teorema 3.1) şi la rezolvarea ecuaţiei operatoriale associate (Theorem 3.2) . Deşi funcţiile "necunoscute" sunt definite implicit, metoda folosită în prezenta lucrare nu foloseşte în demonstraţii Teorema funcţiei implicite, ci permite "construcţia" soluţiei pe baza structurilor bazate pe relaţii de ordine corespunzătoare. Rezultate asemănătoare au fost publicate în [30], [25], [26], [27]. Cazul unor funcţii complexe este abordat în [28] şi continuă în [29].

In Section 2 of the present work we state improved versions of some results first published in [30], only proving the most important assertion from what have been added. Precisely, we show that the analyticity of the function g around its minimum point, is a sufficient condition for the differentiability of the unknown function f at the same point (see Theorem 2.1). We also recall the corresponding general theorem which solves similar operatorial equations (Theorem 2.2). In Section 3, we apply the general results from Section 2, to a concrete functional equation (Theorem 3.1) and to the corresponding operatorial equation (Theorem 3.2). Although the unknown functions are defined implicitely, the method used in the present work does not appeal to the implicit function Theorem, but make possible the direct construction of the solution, using suitable order structures. Similar results were published in [30], [25], [26], [27]. The case of complex functions is approached in [28] and is continued in [29].

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1. Introduction

Obviously, the equation $g = g \circ f$, where g is given, while f is the unknown function, always has the trivial solution $f(x) = x$, $\forall\, x \in A$. Under some assumptions on *g*, there exists a unique decreasing solution of the same equation, which has many supplementary general qualities (see Theorem 2.1). For concrete functions *g*, one obtain special qualities of the corresponding solutions *f*.

In the present work, we recall improved versions of some general known results related to such functional and operatorial equations, which allow the "construction" of the solutions. Then we apply these general-type results, to a concrete functional equation and to the corresponding operatorial equation. In the operatorial case (Theorem 3.2), the solution F is a function of $U \in A \subset X \subset \mathcal{A}(H)$, where X is the commutative algebra of self-adjoint (bounded) operators on an arbitrary complex or real Hilbert space *H*, constructed in [7], pp. 303-305 (here $\mathcal{A}(H)$ is the real ordered space of all self-adjoint operators on *H*). We essentially use the fact that X is also an order-complete vector lattice, with respect to the natural order relation on $\mathcal{A}(H)$.

This paper is directly related to some earlier results: [13], [30], [25], [26], [27]. The study of the same type of equation, but for complex functions, is approached in [28] and continued in [29].

For general-type results partially used in this work see [1]-[12], [14]-[24], [31]-[40].

2. General-type results on the equation $g = g \circ f$, *f* decreasing function or operator

We start by stating an improved version of Theorem 1.1 [30]. The novelty is the point (viii), which was not proved up to now.

2.1. Theorem. *Let* $u, v \in \overline{\mathbf{R}}$, $u < v$, $a \in]u, v[$, $g :\in]u, v[\to \mathbf{R}$ *be a continuous function. Assume that there exist*

(a) $\lim_{x \downarrow u} g(x) = \lim_{x \uparrow v} g(x) = \lambda \in \overline{\mathbf{R}}$ *;*

(b) g *"decreases" (strictly) in the interval* $]u, a]$ *and is strictly increasing in the interval* $[a, v[$.

Then there exists a strictly decreasing function $f :]u, v[\to]u, v[$*, such that*

$$g(x) = g(f(x)) \quad \forall\, x \in]u, v[\qquad (1)$$

and f has the following qualities:

(i) $\lim_{x \downarrow u} f(x) = v, \quad \lim_{x \uparrow v} f(x) = u$ *;*

(ii) *a is the unique fixed point of* f;

(iii) $f^{-1} = f$ in $]u,v[$;

(iv) f *is continuous in* $]u,v[$;

(v) *if* $g \in C^n(]u,v[\setminus \{a\})$, $n \in \mathbf{N} \bigcup \{\infty\}$, $n \geq 1$, *then* $f \in C^n(]u,v[\setminus \{a\})$;

(vi) *if* g *is derivable in* $]u,v[\setminus \{a\}$, *so is* f;

(vii) *if* $g, f \in C^1(]a-\varepsilon, a+\varepsilon[)$ *(for an* $\varepsilon > 0$*), then* $f'(a) = -1$;

(viii) *if* g *is analytic at a, then f is derivable at a and* $f'(a) = -1$;

(ix) *if* $g \in C^3(]u,v[$, $g''(a) \neq a$, *and there exist* $\rho_1 := \lim_{x \to a} f'(x)$, $\rho_2 := \lim_{x \to a} f''(x) \in \mathbf{R}$, *then* $f \in C^2(]u,v[\bigcap C^3(]u,v[\setminus \{a\})$ *and*

$$f''(a) = -\frac{2}{3}\frac{g^{(3)}(a)}{g''(a)} ;$$

(x) let $g_l := g|_{]u,a]}$, $g_r := g|_{[a,v[}$; *then we have the following constructive formulae for f*:

$$f(x_0) = (g_r^{-1} \circ g_l)(x_0) = \sup\{x \in [a,v[;\ \ g_r(x) \leq g_l(x_0)\},\ \forall x_0 \in]u,a] ;$$

$$f(x_0) = (g_l^{-1} \circ g_r)(x_0) = \inf\{x \in]u,a];\ \ g_l(x) \leq g_r(x_0)\},\ \forall x_0 \in [a,v[.$$

Proof. The proof is similar to that of Theorem 1.1 [30], where a geometric meaning of the definition and construction of f is given in Fig. 1. We only have to prove (viii), which represents a sufficient (and not necessary) condition for the differentiability of f at *a*. So, let g be a C^∞-real differentiable function in an interval $]a-\varepsilon, a+\varepsilon[$, which is equal to the sum of its Taylor series around *a*. Then (1) and (ii) lead to:

$$\begin{aligned} g(x) &= g(a) + \frac{g''(a)}{2!}(x-a)^2 + \ldots + \frac{g^{(n)}(a)}{n!}(x-a)^n + \ldots = \\ &= g(f(x)) = g(a) + \frac{g''(a)}{2!}(f(x)-f(a))^2 + \\ &+ \ldots + \frac{g^{(n)}(a)}{n!}(f(x)-f(a))^n + \ldots,\quad \forall x \in]a-\varepsilon, a+\varepsilon[\end{aligned} \tag{2}$$

(note that $g'(a) = 0$ since a is a minimum point of g).

Let $k \geq 2$ be smallest integer for which $f^{(k)}(a) \neq 0$. The conditions on g lead easily to the fact that k is an even number.

From (2), one obtains:

$$\left(\frac{f(x)-f(a)}{x-a} \right)^k = \frac{g_{(a)}^{(k)} + \frac{g_{(a)}^{(k+1)}}{k+1}(x-a) + \ldots}{g_{(a)}^{(k)} + \frac{g_{(a)}^{(k+1)}}{k+1}(f(x)-f(a)) + \ldots} . \tag{3}$$

On the other hand, the facts that f is strictly decreasing and $f(a) = a$, lead to the conclusion

$$\frac{f(x)-f(a)}{x-a} < 0 , \quad \forall x \in]a-\varepsilon, a+\varepsilon[\setminus \{a\} .$$

Thus from (3) one obtains

$$\lim_{x \to a} \left| \frac{f(x)-f(a)}{x-a} \right| = \lim_{x \to a} - \frac{f(x)-f(a)}{x-a} = 1 \text{ , i.e.}$$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = -1 \text{ , which proves (viii).}$$

Note also that from "f is strictly decreasing", (i) and (ii), we infer that f applies $]u,a]$ onto $[a,v[$ and $[a,v[$ onto $]u,a]$ (this will be used in the proof of Theorem 3.2).

■

Next we recall the abstract operatorial version of solving the same type of equations.

In the following, X will be an arbitrary order-complete vector lattice, and $\text{Izom}_+(X)$ will be the set of all vector space isomorphisms $T : X \to X$ which apply X_+ onto X_+ (X_+ is the cone of positive elements of X).

2.2. Theorem. *Let X, X_+ be as above, $a \in X$, A_l a convex subset such that*

$$a \in A_l \subset \{x \in X; \quad x \leq a\} ,$$

A_r *a convex subset such that*

$$a \in A_r \subset \{x \in X;\ \ x \geq a\}\ .$$

Let $g_l : A_l \to X$ *be a convex operator such that*

$$\partial g_l(x) \bigcap (-\mathrm{Izom}_+(X)) \neq \Phi, \quad \forall x \in A_l \setminus \{a\}\ .$$

Let $g_r : A_r \to X$ *be a convex operator such that*

$$\partial g_r(x) \bigcap \mathrm{Izom}_+(X) \neq \Phi, \quad \forall x \in A_r \setminus \{a\}\ .$$

($\partial g(x)$ *is the set of all subdifferentials of* g *at* x).

Assume that $g_l(a) = g_r(a)$ *and* $R(g_l) = R(g_r)$, *where* $R(g)$ *is the range of* g.

Let $g : A := A_l \bigcup A_r \to X$ *be defined by*

$$g(x) := \begin{cases} g_l(x), & x \in A_l, \\ g_r(x), & x \in A_r. \end{cases}$$

Then there exists a strictly decreasing map $F : A \to A$ *such that*

$$g(x) = g(F(x)), \quad \forall\, x \in A \tag{1'}$$

and F has the following qualities:

(i) *a is the only fixed point of F;*

(ii) *there exists* $F^{-1} : A \to A$ *and* $F^{-1} = F$ *;*

(iii) *the following "constructive formulae" for* $F(x_0)$ *hold:*

$$F(x_0) = (g_r^{-1} \circ g_l)(x_0) = \sup\{x \in A_r;\ \ g_r(x) \leq g_l(x_0)\},\ \forall\, x_0 \in A_l\ ;$$

$$F(x_0) = (g_l^{-1} \circ g_r)(x_0) = \inf\{x \in A_l;\ \ g_l(x) \leq g_r(x_0)\},\ \forall\, x_0 \in A_r\ .$$

For the proof of this Theorem see [30], Theorem 1.10, pp. 72-74.

3. Applications

We start by an application of Theorem 2.1, to a function g which is convex and analytic in the whole interval $]0,\infty[$, namely to the function $g(x) := \exp(x) + e \cdot x^{-1}$, $x > 0$. This will imply the differentiability of the solution f at $a = 1$, the minimum point of g, where generally, (when g is not analytic), the study of derivability can be a problem.

3.1. Theorem. *There exists a strictly decreasing function* $f:]0,\infty[\to]0,\infty[$ *such that*

$$e^x + \frac{e}{x} = e^{f(x)} + \frac{e}{f(x)}, \quad \forall x > 0 \tag{1''}$$

and f has the following qualities:

(i) $\lim_{x\downarrow 0} f(x) = \infty, \quad \lim_{x\uparrow\infty} f(x) = 0$*;*

(ii) $a = 1$ *is the unique fixed point of* f*;*

(iii) $f^{-1} = f$ in $]0,\infty[$;

(iv) $f \in C^1(]0,\infty[) \cap C^\infty(]0,\infty[\setminus\{1\})$ *and* $f'(1) = -1$*;*

(v) *if there exists* $\lim_{x\to 1} f''(x) \in \mathbf{R}$*, then*

$$f \in C^2(]0,\infty[) \cap C^3(]0,\infty[\setminus\{1\})$$

and $f''(1) = \frac{10}{9}$ *; in particular, f is convex in a neighbourhood of* 1*;*

(vi) *if there exists* $\lim_{x\to 1} f''(x) \in \mathbf{R}$*, then for* $\varepsilon > 0$ *sufficiently small, we have* $f(x) + x \geq 2$*,* $\forall x \in]1-\varepsilon, 1+\varepsilon[$*, and equality holds if and only if* $x = 1$*;*

(vii) *the following "constructive" formulae for* $f(x_0)$ *hold:*

$$f(x_0) = (g_r^{-1} \circ g_l)(x_0) = \sup\{x \in [1,\infty[;\ e^x + e\cdot x^{-1} \leq e^{x_0} + e\cdot x_0^{-1}\}, \quad \forall x \in]0,1[,$$

$$f(x_0) = (g_l^{-1} \circ g_r)(x_0) = \inf\{x \in]0,1];\ e^x + e\cdot x^{-1} \leq e^{x_0} + e\cdot x_0^{-1}\}, \quad \forall x \in [1,\infty[;$$

(viii) *for any integer* $n \geq 1$*, we have* $f(x) \in \mathbf{Z} \Leftrightarrow n = 1$.

Proof. One applies Theorem 2.1 to $]u,v[=]0,\infty[$, $g(x) := e^x + e\cdot x^{-1}$ $x > 0$. Obviously, we have $\lim_{x\downarrow 0} g(x) = \lim_{x\uparrow\infty} g(x) = \infty$. On the other hand, g is analytic in $]0,\infty[$ and $g'(x) = e^x - \frac{e}{x^2} = \frac{x^2e^x - e}{x^2}$, $\forall x > 0$; hence $g'(x) < 0$ for $x \in]0,1[$, $g'(x) = 0$ for $x = 1$, $g'(x) > 0$ for $x > 1$. We also have $g''(x) = e^x + \frac{2e}{x^3} > 0$, $\forall x > 0$, hence g is strictly convex. Thus, in particular, conditions (a) and (b) on g are accomplished (where $a = 1$), so that, from Theorem 2.1 we infer that there exists a strictly decreasing function $f:]0,\infty[\to]0,\infty[$ such that (1″) and (i) – (iv) of the present Theorem hold. To prove (v), assume that $\lim_{x\to 1} f''(x)$ does exist in **R**. Then, by Theorem 2.1, we have:

$$f''(1) = -\frac{2}{3}\frac{g^{(3)}(1)}{g''(1)} = -\frac{2}{3}\cdot\frac{-5e}{3e} = \frac{10}{9} > 0 \ .$$

Thus (v) is proved, while (vi) follows from (v). Precisely, $f \in C^{\infty}(]0,\infty[) \setminus \{1\})$ and the hypothesis of the existence in $\mathbf{R}$ of the limit $\lim_{x\to 1} f''(x)$ lead to $f \in C^2(]0,\infty[)$. Thus, the positivity of f'' at $a = 1$ leads to the positivity of f'' on an interval $]1-\varepsilon,\ 1+\varepsilon[$. Hence f is strictly convex in $]1-\varepsilon,\ 1+\varepsilon[$ and, the well known property of such differentiable functions yields:

$$f(x) \geq f(1) + f'(1)(x-1) \overset{(i),(iv)}{=} 1 + (-1)(x-1) = 2 - x \Leftrightarrow f(x) + x \geq 2\ ,$$

$\forall x \in]1-\varepsilon,\ 1+\varepsilon[$, the inequality being strictly for $x \neq 1$. "Formulae" mentioned at (vii) are direct applications of those from Theorem 2.1, (x), to our function *g*. It remains to prove (viii). If $f(n) = m \in \mathbf{Z}$ for an integer $n \geq 1$, since f has positive values, we have $m \in \mathbf{Z}$, $m \geq 1$. Because of (1"), one obtains

$$e^n + \frac{e}{n} = e^m + \frac{e}{m}\ , \quad \text{i.e.}$$

$$mne^n - mne^m + me - ne = 0\ ,$$

or, equivalently,

$$mne^{n-1} - mne^{m-1} + m - n = 0\ . \qquad (4)$$

If $m \neq n$, $m, n \geq 1$, then at least one of the integers m, n is greater or equal to 2, so that (4) sais that *e* is a root of the algebraic equation with integer coefficients:

$$mnx^{n-1} - mnx^{m-1} + m - n = 0\ .$$

This contradicts the transcendency of *e*. Thus, we must have $m = n$, hence $f(n) = n$, which implies (via (ii)) $n = 1$. Conversely, for $n = 1$, $f(n) = f(1) \overset{(ii)}{=} 1 = n$. Now the proof is complete.

■

Next we go on to the operatorial equation related to (1"). Some special care is inquired during the following proof, in order to show that $g(U) := \exp(U) + eU^{-1}$ satisfies all the conditions mentioned in the statement of Theorem 2.2.

Let H be an arbitrary Hilbert space. Let $\mathcal{A}(H)$ be the real vector space of all self-adjoint (linear bounded) operators acting on *H*. Let $B \in \mathcal{A}(H)$ be a fixed operator. One defines

$$\mathcal{A}_1 = \mathcal{A}_1(B) := \{U \in \mathcal{A}(H); UB = BU\}\ ,$$

$$X = X(B) := \{U \in \mathcal{A}_1;\ UV = VU \quad \forall V \in \mathcal{A}_1\}$$

$$X_+ := \{U \in X;\ \langle U(h), h\rangle \geq 0 \quad \forall h \in H\}.$$

It is known that X is an order-complete vector lattice and a commutative algebra of operators (see [7], pp. 303-305).

3.2. Theorem. *Consider the following convex subsets of the space* $X = X(B)$ *defined above:*

$$A_l := \{U \in X;\ \sigma(U) \subset]0,1[\} \cup \{I\},$$

$$A_r := \{U \in X;\ \sigma(U) \subset]1,\infty[\} \cup \{I\},$$

where $\sigma(U)$ *is the spectrum of* U *and* I *is the identity operator. Let* $A := A_l \cup A_r$. *Then there exists a strictly decreasing map* $F : A \to A$ *such that*

$$\exp(U) + e\cdot U^{-1} = \exp(F(U)) + e\cdot (F(U))^{-1},\ \forall U \in A. \qquad (1''')$$

The map F has the following additional qualities:

(i) *I is the unique fixed "point" of F;*

(ii) *F is invertible and $F^{-1} = F$ in A;*

(iii) *F can be "constructed" using the "formulae":*

$$F(U_0) = \sup\{U \in A_r;\ \exp(U) + eU^{-1} \leq \exp(U_0) + e\cdot U_0^{-1}\},\ U_0 \in A_l,$$

$$F(U_0) = \inf\{U \in A_l;\ \exp(U) + eU^{-1} \leq \exp(U_0) + e\cdot U_0^{-1}\},\ U_0 \in A_r.$$

Proof. Let

$$g : D := \{U \in X;\ \sigma(U) \subset]0,\infty[\} \to D,\ g(U) := \exp(U) + e\cdot U^{-1}.$$

Obviously, D is a convex set, which contains both convex subsets A_l, A_r. We will prove that g is convex in D, which will imply the convexity of $g_l := g|_{A_l}$ and $g_r := g|_{A_r}$. Since it was proved that $U \mapsto U^n$, $n \in \mathbf{Z}_+$ is convex in $X_+ \supset D$, (see [30]), it follows easily that

$$\exp(U) = \sum_{n=0}^{\infty} \frac{U^n}{n!} = \sup_{n\in\mathbf{Z}_+} \sum_{k=0}^{n} \frac{U^k}{k!}$$

is convex as supremum of convex operators. So, it is sufficient to prove that $U \mapsto U^{-1}$ is convex in D, i.e.

$$((1-\lambda)U_1 + \lambda U_2)^{-1} \leq (1-\lambda)U_1^{-1} + \lambda U_2^{-1},\ \forall U_1, U_2 \in D,\ \forall \lambda \in]0,1[. \qquad (5)$$

Since U_1, U_2 are positive, invertible, permutable self-adjoint operators, the last relation (5) is equivalent to

$$((1-\lambda)I+\lambda U_1^{-1}U_2)^{-1}\le(1-\lambda)I+\lambda U_1U_2^{-1},\ \ U_1,U_2\in D,\ \ \lambda\in]0,1[\ ,$$

i.e.

$$((1-\lambda)I+\lambda T)^{-1}\le(1-\lambda)I+\lambda T^{-1},\ \ \forall T\in D,\ \forall\lambda\in]\,0,1\,[\,. \tag{5'}$$

Thus we have "simplified" the problem from two operators U_1,U_2, to one operator, $T:=U_1^{-1}U_2\in D$. From the elementary inequality

$$((1-\lambda)\cdot 1+\lambda t)^{-1}\le(1-\lambda)\cdot 1+\lambda t^{-1},\ \ \forall t>0,\ \forall\lambda\in]\,0,1\,[$$

(the convexity of $t\mapsto t^{-1}$ is elementary on $]\,0,\infty\,[$), by integration on the spectrum $\sigma(T)\subset]\,0,\infty\,[$, with respect to the spectral measure dE_T attached to T, one obtains:

$$((1-\lambda)I+\lambda T)^{-1}=\int_{\sigma(T)}((1-\lambda)\cdot 1+\lambda t)^{-1}dE_T\le\int_{\sigma(T)}((1-\lambda)\cdot 1+\lambda t^{-1})dE_T$$
$$=(1-\lambda)I+\lambda T^{-1},\quad T\in D,\quad \lambda\in]0,1[\,.$$

Thus (5'), hence (5), are proved. It follows that g is convex in D. Let $g_l:=g|_{A_l}$. Then

$$g_l'(U)(V)=(\exp(U)-eU^{-2})V\le 0\quad \forall U\in A_l\setminus\{I\},\ \forall V\in X_+$$

as a product of two permutable self-adjoint operators, which verify, $\exp(U)-eU^{-2}<0$, $V\ge 0$ ($U\in A_l\setminus\{I\}$ implies $\sigma(U)\subset]\,0,1[$ and we have seen that $g'(t)=\exp(t)-et^{-2}<0\quad \forall t\in]\,0,1[$; this leads (via functional calculus) to $\sigma(\exp(U)-e\cdot U^{-2})=\sigma(g'(U))=g'(\sigma(U))\subset]-\infty,0[$ and further $\sigma[(\exp(U)-e\cdot U^{-2})^{-1}]\subset]-\infty,0[$) . The conclusion is that for any $U\in A_l\setminus\{I\}$, $g'(U)\in-\mathrm{Izom}_+(X)$. Similarly, for $U\in A_r\setminus\{I\}$, $g'(U)\in\mathrm{Izom}_+(X)$. It remains to verify that $R(g_l)=R(g_r)$. Let $g_l(U_1)\in R(g_l)$. Let $U_2:=F(U_1)$, where F is associated with the real function f from Theorem 3.1 by Lemma 3.3.1 [6] (functional calculus; see also [12]). Then $\sigma(U_2)=\sigma(F(U_1))=f(\sigma(U_1))\subset]1,\infty[$, since f applies $]\,0,1[$ onto $]1,\infty[$ and $\sigma(U_1)\subset]\,0,1[$. Thus $U_2\in A_r\setminus\{I\}$. We next prove that $g_l(U_1)=g_r(U_2)$. Let $t_1\in\sigma(U_1)$. By Theorem 3.1, we have

$$g_l(t)=g_r(f(t))\quad \forall t\in]\,0,1[\supset\sigma(U_1)\ .$$

By integration on $\sigma(U_1)$ with respect to the spectral measure dE_{U_1}, the last relation leads to

$$g_l(U_1)=\int_{\sigma(U_1)}g_l(t)dE_{U_1}=\int_{\sigma(U_1)}g_r(f(t))dE_{U_1}=$$
$$=g_r(F(U_1))=g_r(U_2)\in R(g_r).$$

The conclusion is $g_l(U_1) \in R(g_r) \quad \forall U_1 \in A_l$, hence $R(g_l) \subset R(g_r)$. Similalrly, one proves that $R(g_r) \subset R(g_l)$, hence the last condition in the statement of Theorem 2.2 is verified. Applying Theorem 2.2 to the operators g_l, g_r obtained from $g(U) = \exp(U) + e \cdot U^{-1}$ by restriction to A_l, A_r, the conclusions (1'''), (i)-(iii) of the present Theorem follow.

■

4. Conclusions

In the first part of this work we have improved the general Theorem 2.1, by adding the assertion (viii). Such-type results are motivated by the fact that most of the elementary functions g are analytic, so that the corresponding solutions f are derivable at the critical point a of g. We also recall the abstract operatorial version which solves such equations (both these general results are based on the same constructive method, allowed by some order structures).

In the second part of this work, (Section 3), we solve a concrete functional equation, (and also the corresponding operatorial equation). Due to the special qualities of g $(g(x) = \exp(x) + e \cdot x^{-1}, \ \ x > 0)$, one obtains special corresponding qualities of f, such as those mentioned at (iv), (vi), (viii) of Theorem 3.1.

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