CONSENSUS CONTROL OF DISCRETE-TIME MULTI-AGENT SYSTEMS

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In this paper has been studied the problem of consensusability for single-input discrete-time MASs over directed communication graphs with fixed topology. The control input of each agent can only use its local state and the states of its neighbors. The results obtained in this paper extend some recent known results for the case when the network topology is a weighted digraph jointly containing a spanning tree.

Keywords: stability matrix, consensus protocol, Linear Discrete-Time Invariant dynamical system, weighted digraph, asymptotic stability.

1. Introduction

Distributed control of systems is of a major interest in recent years. Dynamic agents that are fulfilling the control tasks in a decentralized manner compose an information flow network and can be placed in the category of Network Control Systems (NCSs). Time delays in information flows introduce stability problems. Design and stability issues of NCSs are discussed in [1], where the NCS's performance is influenced by a key scheduling restriction as a measure of the network-induced delays. An integrative approach to construct and operate these systems is given by their representation as Multi-agent Systems (MAS) architectures. Beside the distributed control, the reasons why MAS have sparked the interest of researchers of NCS, are represented by a multitude of problems that could be solved, among which we mention: collaborative leading, consensus decision, programming and planning. In terms of control systems, MAS will facilitate the development of adaptive, reconfigurable, responsive and flexible automation systems [2].

This paper discusses the consensus problems for a set of discrete-time agents with directed network information flow under fixed communication topology. The information exchange among agents is viewed as a communication

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graph and, therefore, modeled using algebraic graph theory, to describe how much information each agent has access to at a specific moment in time.

The coordination control for networks of dynamical agents using distributed consensus (i.e. making the states of all agents identical through local information exchange) has attracted researchers from various disciplines. Various models have been proposed and investigated, for both discrete-time or continuous time consensus models, among the first being [3], [4] and [5]. Among the relative recent related works, we mention [6], where the author investigates the problem when discrete-time MAS are consensusable under undirected graph and [7] where the authors investigate the consensus problem with infinite time-varying delays for linearly coupled static network. Various extensions of the problems when the discrete-time multiagent dynamic systems are consensusable on switching topologies with communication delays, and in the presence of system parameter variations are presented in [8]. Ma and Zhang give the necessary condition of consensusability of MASs with fixed topology and agents described by linear time-invariant systems with respect to a set of admissible consensus protocols [9]. A unified way to obtain the consensus of multiagent systems with a time-invariant communication topology and the synchronization of complex networks is addressed in [10]. The joint effects of agent dynamic and network topology on the consensusability of linear discrete-time MAS are investigated in [11].

2. Preliminaries

2.1. Problem formulation

Let consider a set of N homogenous dynamic agents described by the following state space model:

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \qquad x_i(k_o) = x_{i0}$$
$$y_i(k) = Cx_i(k), \qquad i = 1,2,...,N \tag{1}$$

where, denoting by classical notation, $x_i(k) \in \Re^n$ is the state vector for each agent i, $x_i(k_o)$ represents the initial conditions, $u_i(k) \in \Re^m$ is the control input vector, $y_i(k) \in \Re^p$ is the measured output vector. $A \in \Re^{n \times n}$ is the state matrix, $B \in \Re^{n \times m}$ is the input matrix and $C \in \Re^{p \times n}$ is the output matrix. Equation (1) represents the state-space description of a finite dimensional Linear Discrete Time Invariant (LDTI) dynamical system. We will consider farther the case when $m=p=1$ corresponding to agents that have single-input and single output (SISO). We consider that all agents have the same dynamical model, therefore their transfer matrix will be identical, i.e. $T(z) = C(zI - A)^{-1}B$. The N agents are connected in a communication network with a topology described by either directed or undirected graph. Further we will consider only directed graphs. The distributed

control system should lead to a steady state based on consensus among the agents. State transitions are influenced by the communications among neighboring agents. Finally, the control algorithms should respect the equation (2).

$$\lim_{k\to\infty}\left\|x_i(k)-x_j(k)\right\|=0 \qquad \forall i,j\in N \tag{2}$$

Let $G=(V,L,A)$ be a directed graph of order n where $V=\{v_1,v_2,...v_n\}$ is the non-empty finite set of nodes (vertices), $L=\{l_1,l_2,...l_n\}$ is the finit set of edges (links) between ordered pairs of nodes $(v_i,v_j,)$ i.e. $L\subseteq V\times V$, and $A=\left[a_{i,j}\right]_{i,j=1,1}^{N,N}$ is a weighted adjacency matrix with nonnegative adjacency elements $a_{i,j}$. A direct edge of G is denoted by l_{ij}. If the weights in the matrix A satisfy the following conditions:

$$a_{i,j}>0 \quad if \quad (l_i,l_j)\in L\,; a_{i,j}=0 \quad if \quad (l_i,l_j)\notin L \tag{3}$$

then the graph G is a weighted directed graph. All the nodes that communicate to v_i compose the set of its neighbors, denoted by $N_i=\{v_j\in V:(v_j,v_i)L\}$.

2.2. Basic definitions.

<u>Consensusability.</u> Let consider a particular state feedback protocol given by:

$$u_i(k)=K\sum_{j=1}^{N}a_{i,j}[x_j(k)-x_i(k)],i=1,2,...,N \tag{4}$$

where $a_{i,j}$ are the elements of the adjacency matrix of the directed graph G (describing the communication links) and $K\in\mathfrak{R}^{m\times n}$ is the constant state feedback gain ($C=I$). *Consensusability* in MAS is reached if there exists a control protocol as in equation (4) that will lead to the condition in equation (2).

<u>Stability.</u> An LDTI system is said to be asymptotically stable if and only if the eigenvalues of the system dynamics matrix A are less than unity in their magnitude, i.e. $\left|\lambda_i<1\right|,i=1,2,...,N$. In this case, matrix A is known as a *stability matrix*.

3. Results on solving the agents consensusability problem

The condition for consensusability using the protocol (4) is known ([8], [9], [11]) and was utilised as the basis to obtain milder conditions for consensusability. In the same line, we will discuss and prove several conditions that guarantee the success of a consensus protocol in MAS. The research is based on the previous results published by Marinovici in his PhD thesis [12], being oriented on a better understanding of the relationship between consensusability of LTI-MASs and the dynamic properties of the MASs. In this aim the proofs of four lemmas which provide the sufficient conditions for reaching consensus among a set of discrete-time agents are presented.

<u>Lemma 1.</u> The consensus in MAS with agents having a discrete-time dynamics modeled by equation (1) can be reached according the protocol described by (4), if and only if there exists a constant control gain $K \in \mathfrak{R}^{m \times n}$, such that $A - \lambda_i BK$ is a stability matrix for any $i = 2,3,...,N$.

<u>Proof.</u> The state equation in (1) can be rewritten based on (4):

$$x_i(k+1) = Ax_i(k) + BK\sum_{j=1}^{N} a_{i,j}[x_j(k) - x_i(k)] = Ax_i(k) - BK\sum_{j=1}^{N} l_{i,j}BKx_j(k) \qquad (5)$$

$l_{i,j}$ are the edges of the directed graph's nodes (communication links of the corresponding agents). $L_G = \left[l_{i,j}\right]_{i,j=1,1}^{N,N}$ is the Laplacian. Because L_G has a zero eiegenvalue and the non-zero eigenvalues can be arranged in ascending order $0 \le |\lambda_1| \le |\lambda_2|... \le |\lambda_N|$, then results:

$$l_{i,i} > 0; l_{i,j} \le 0, \forall i \ne j; \sum_{j=1}^{N} l_{i,j} = 0, i = 1,2,...,N.$$

Gathering all the agents' states in an unique vector $X(k)$ defined as the overall state vector af the system: $X(k) = \left[x_1^T(k) \ x_2^T(k) \ ... \ x_N^T(k)\right]^T \in \mathfrak{R}^{nN \times 1}$, the overall state dynamics is given by:

$$X(k+1) = \begin{bmatrix} A & 0 & ... & 0 \\ 0 & A & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & A \end{bmatrix} \cdot X(k) + \begin{bmatrix} (\sum_{j=1}^{N} a_{1,j})BK & -a_{1,2}BK & ... & -a_{1,N}BK \\ -a_{2,1}BK & (\sum_{j=1}^{N} a_{2,j})BK & ... & -a_{2,N}BK \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N,1}BK & -a_{N,2}BK & ... & (\sum_{j=1}^{N} a_{N,j})BK \end{bmatrix} \cdot X(k)$$

$$(6)$$

Invoking the definition of Kronecker product, equation (6) can be reduced:

$$X(k+1) = [I_N \otimes A - L \otimes BK]X(k) \qquad (7)$$

where $L \in \mathfrak{R}^{N \times N}$ is the Laplacian matrix of the digraph G.

So one can affirm that protocol (4) solves the consensus problem if the states of system (5) satisfy the condition (2) q.e.d.

<u>Remark.</u> In the case of a communication network denoted by a directed graph G that has a directed spanning tree, the proposed protocol can work successfully if and only if *A-BK* and *A- $\lambda_i FC$* are Hurwitz. *F* is a stable filter of size *m* x *m* and λ_i, *i=1,2,..,N* are non-zero eigenvalues of the Laplacian matrix L_G. The proof of this statement is done in [6].

<u>Definition</u>**.** The disagreement vector *d(k)* is defined by:

$$d(k) = X(k) - [(1r^T) \otimes I_n]X(k) \in \mathfrak{R}^{nN \times 1} \qquad (8)$$

where $r^T = [r_1\, r_2 ... r_n] \in \Re^{1 \times N}$ is the left eigenvector of the Laplacian matrix L associated with the zero eigenvalue (i.e. $r^T L = 0$), which satisfy the condition $r^T \mathbf{1} = 1$.

Based on the dynamics in equation (7), the disagreement dynamics becomes:

$$d(k+1) = (I_n \otimes A - L \otimes BK)X(k) - [(1r^T) \otimes I_n](I_n \otimes A - L \otimes BK)X(k) \quad (9)$$

<u>Lemma 2</u> Solving the agents consensusability problem is equivalent to solving the asymptotic stability problem for the disagreement dynamics.

<u>Proof.</u> First let consider (8) written in a different form:

$$d(k) = [I_N \otimes I_N - (1r^T) \otimes I_n]X(k) = [(I_N - 1r^T) \otimes I_n]X(k) = (\hat{M} \otimes I_N)X(k) \quad (10)$$

where

$$\hat{M} = \begin{bmatrix} 1-r_1 & -r_2 & ... & -r_N \\ -r_1 & 1-r_2 & ... & -r_N \\ \vdots & \vdots & \ddots & \vdots \\ -r_1 & -r_2 & ... & 1-r_N \end{bmatrix}$$

The matrix $\hat{M}$ can be rewrite using the transformation matrix $\hat{M}_T$,

$$\hat{M}_T = \begin{bmatrix} 1 & 0 & ... & -1 \\ 0 & 1 & ... & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \end{bmatrix}$$

So that:

$$\hat{M}_T \hat{M} \hat{M}_T^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -r_1 & -r_2 & \cdots & -r_{N-1} & 0 \end{bmatrix}$$

Thus, it is easy to see that 0 is a simple eigenvalue for $\hat{M}$ and 1 is an eigenvalue of multiplicity N-1. Moreover, 1 is the right eigenvector corresponding to the 0 eigenvalue. Therefore, according to equation (10), $d(k)=0$ if and only if all the values $x_i(k)$ are equal. In other words, the consensus problem is solved if and only if $\lim_{k \to \infty} d(k) = 0$ (q.e.d.)

In the following we discuss the asymptotic stability of the disagreement dynamics (9).

<u>Lemma 3</u> The asymptotic stability of a MAS which various state transformations is ensured if and only if the associated transfer matrices are stability matrices.

<u>Proof.</u> First we define the transformation matrix $T \in \mathfrak{R}^{N \times N}$, $T = [1 \ Y]$ with Y the output vector, which can be deduced from the diagreement vector by:

$$\tau(k) = (T^{-1} \otimes I_n)d(k) = [\tau_1^T \quad \tau_2^T \quad ... \quad \tau_N^T] \tag{11}$$

The state transformation leads to a new dynamics:

$$\tau(k+1) = (I_n \otimes A - J \otimes BK)\tau(k) \tag{12}$$

with $T^{-1} = \begin{bmatrix} r^T \\ W \end{bmatrix}$, $W \in \mathfrak{R}^{(N-1) \times N}$,

$$J = T^{-1}LT = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}, H = \begin{bmatrix} \lambda_2 & 0 & ... & * \\ 0 & \lambda_3 & ... & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_N \end{bmatrix}$$

From the combination of (10) and (11) it results:

$$\tau_1(k) = (r^T \otimes I_n)(\hat{M} \otimes I_N)X(k) = 0 \tag{13}$$

and the equation (12) can be expanded as:

$$\tau(k+1) = \begin{bmatrix} A - \lambda_1 BK & 0 & ... & 0 \\ 0 & A - \lambda_2 BK & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & A - \lambda_N BK \end{bmatrix}\tau(k), \quad with \quad \lambda_1 = 0 \tag{14}$$

Thus, asymptotic stability of the transformations $\tau_i(k), i = 2,...,N$ is ensured if and only if all the subsystems of MAS which respect:

$$\tau_i(k+1) = (A - \lambda_i BK)\tau_i(k), i = 2,...,N \tag{15}$$

Are asymptotic stable, which is equivalent with $(A - \lambda_i BK), i = 2,...,N$ matrices being stability matrices (q.e.d).

<u>Lemma 4</u> If a set of N discret-time agents, having each the same LTDI model (1) is interconnected through a network with a communication topology represented by the direct graph G has a feedback control protocol (4) which satisfies Lemma 1, then the condition

$$x_i(k) \xrightarrow{k \rightarrow \infty} (r^T \otimes A^k)\begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_N(0) \end{bmatrix}, i = 2,...,N \tag{16}$$

is also satisfied.

Proof.

According to (7), the solution to the agents' dynamics is given by:

$$X(k) = [I_N \otimes A - L \otimes BK]^k X(0).$$

As

$$I_N \otimes A - J \otimes BK = \begin{bmatrix} A & 0 \\ 0 & I_N \otimes A - H \otimes BK \end{bmatrix} \text{ and}$$

$T \otimes I_N = (T^{-1} \otimes I_N)^{-1}$ it results:

$$X(k) = (T \otimes I_N) = \begin{bmatrix} A^k & 0 \\ 0 & [I_N \otimes A - H \otimes BK]^k \end{bmatrix} (T^{-1} \otimes I_N) X(0) \tag{17}$$

From Lemma (1), $I_N \otimes A - H \otimes BK$ is a stability matrix and therefore

$$[I_N \otimes A - H \otimes BK]^k \xrightarrow{k \to \infty} 0.$$

That leads to:

$$X(k) \xrightarrow{k \to \infty} T \otimes I_N \begin{bmatrix} A^k & 0 \\ 0 & 0 \end{bmatrix} (T^{-1} \otimes I_N) = [(1 \cdot r^T) \otimes A^k] X(0) \tag{18}$$

Equation (18) confirms the validity of the hypothesis (16) (q.e.d).

Remark. According Lemma 4 if A is a stability matrix (i.e. all eigenvalues are placed inside the unit circle), one can always reach the consensus value.

4. Conclusions

The paper analyses some issues of the consensus control decision for discrete-time multi-agent dynamic Networked Control Systems. The research is limited to NCSs/MASs with communication topology represented by directed graphs and derives the consensusability condition for single-input dynamic agent systems under state feedback. A group of four Lemmas show how to synthesize the consensus feedback control law when the consensusability condition holds.

The statements of the four Lemmas provides a sufficient condition for reaching consensus among a set of discrete-time agents and lead to some interesting conclusions regarding the consensus values with respect to the agents' dynamics. Thus, if matrix A is a stability matrix, then the consensus value reached by the agents is going to be 0. If matrix A has all eigenvalues outside the unit circle, the consensus is going to be reached asymptotically at infinity. The case when matrix A has eigenvalues on the unit circle proves to be the critical case for consensus of agents under the protocol (4) to happen at a constant value.

Our future work will study the consensus problem for delayed coupled systems under a more general case where time delays between different nodes can be different.

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R E F E R E N C E S

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