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ON THE CHARACTERIZATION OF GENERALIZED DUAL FRAMES

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In this paper, we generalize a new concept of duality for frames by a left invertible operator and introduce alternate dual frames corresponding to this operator for both discrete and continuous frames. Several characterizations of alternate dual frames are established to obtain more reconstruction formulas. Finally, we survey the robustness of these dual frames under some perturbations.

Keywords: Frames, dual frames, continuous frames; A-dual frames.

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1. Introduction and preliminaries

The theory of frames dates back to Gabor [16] and Duffin and Schaeffer [15]. However, the frame theory had not attracted much attention until 1986. So far, the theory of frames has been growing rapidly. Many researchers have seen great achievements in pure mathematics, science, and engineering such as image processing, signal processing, sampling and approximation theory [2, 9, 12, 23]. A basic problem of interest in connection with theory of frames is about the characterization of dual frames of a given frame [4, 6, 8, 11]. Since, it is usually complicated to calculate a dual frame explicitly [10, 19], the idea of looking a new concept of dual frames has appeared in several papers, for example pseudo-dual, approxi-mately dual and generalized dual frames are various concepts of duality [12, 14]. Generalized duals with corresponding invertible operators are considered by many researchers [14, 18]. Inspired by this generalization, we introduce the concept of A-duality correspondence to left invertible operator A. The main subject of this paper deals with the characterization of A-dual frames for both discrete and continuous frames. More reconstruction formulas are obtained by using these duals. For the rest of this section we briefly recall the definition and properties of discrete and continuous frames.

A sequence $\Phi = \{\phi_i\}_{i\in I}$ where I is a finite or countable index set, is called a *frame* in a Hilbert space $\mathcal{H}$ if there are constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} |\langle f, \phi_i\rangle|^2 \leq D\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1}$$

If Φ satisfies the right hand of (1), then Φ is called a *Bessel sequence*. The constants C and D are called *frame bounds*. The supremum of all lower frame bounds is called the *optimal lower frame bound* and the infimum of all upper frame bounds is called the *optimal upper frame bound*. For a Bessel sequence $\Phi = \{\phi_i\}_{i\in I}$, the *synthesis operator* or *pre-frame operator* $T_\Phi : \ell^2 \to \mathcal{H}$ is defined by $T_\Phi\{c_i\}_{i\in I} = \sum_{i\in I} c_i\phi_i$. The *analysis operator* is the adjoint of T_Φ and it is given by $T^*_\Phi f = \{\langle f, \phi_i\rangle\}_{i\in I}$. The operator $S_\Phi = T_\Phi T^*_\Phi$, so called

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the *frame operator* is self adjoint, positive and invertible when Φ is a frame in $\mathcal{H}$ [11]. The invertibility of S_Φ gives the reconstruction formula

$$f = S_\Phi^{-1} S_\Phi f = \sum_{i\in I} \langle f, \phi_i \rangle S_\Phi^{-1}\phi_i = \sum_{i\in I} \left\langle f, S_\Phi^{-1}\phi_i \right\rangle \phi_i. \tag{2}$$

A Bessel sequence $\Psi = \{\psi_i\}_{i\in I}$ in $\mathcal{H}$ is called a *dual* for a Bessel sequence $\Phi = \{\phi_i\}_{i\in I}$ if

$$f = \sum_{i\in I} \langle f, \psi_i \rangle \phi_i = T_\Phi T_\Psi^* f. \tag{3}$$

Note that $\left\{S_\Phi^{-1}\phi_i\right\}_{i\in I}$, which is a dual of Φ, is called *the canonical* dual frame [11].

Definition 1.1. *A Riesz basis for $\mathcal{H}$ is a sequence of the form $\{Ue_i\}_{i\in I}$, where $\{e_i\}_{i\in I}$ is an orthonormal basis in $\mathcal{H}$ and $U \in B(\mathcal{H})$ is a bijective operator.*

Let $\mathcal{H}$ be a complex Hilbert space and (X, ν) a measure space with positive Radon measure ν. A mapping $F : X \to \mathcal{H}$ is called a *continuous frame* with respect to (X, ν), if

(1) F is weakly measurable;
(2) there exist constants $C, D > 0$ such that

$$C\|f\|^2 \leq \int_X |\langle f, F(x)\rangle|^2 d\nu(x) \leq D\|f\|^2, \quad (f \in \mathcal{H}). \tag{4}$$

The mapping F is called *Bessel* if the second inequality in (4) holds. For more properties of continuous frames see [1, 17, 21]. Suppose that F is Bessel, then the operator $T_F : L^2(X) \to \mathcal{H}$ defined by

$$T_F\xi = \int_X \xi(x)F(x)d\nu(x)$$

is a bounded linear operator; so called *the synthesis operator*. Also, its adjoint given by

$$(T_F^*\xi)(x) = \langle \xi, F(x)\rangle, \quad x \in X,$$

is called *the analysis operator* of F. The *continuous frame operator*, which is invertible, positive as well as self adjoint, is defined by $S_F = T_F T_F^*$. Let F be a continuous frame with frame bounds C and D respectively. The mapping $S_F^{-1}F$, which is a continuous frame with the bounds $1/D$ and $1/C$, is called *the canonical dual* and denoted by F^c. Every Bessel mapping $G : X \to \mathcal{H}$ that satisfies the following

$$f = \int_X \langle f, F(x)\rangle G(x)\, d\nu(x),$$

is called a dual for F, i.e. $S_{G,F} := T_G T_F^* = I$. A continuous frame is called Riesz type if it has only one dual.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}(\mathcal{H})$ are the collections of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $\mathcal{H}$, respectively. $\mathcal{B}^l(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$ with a left inverse. We denote the identity operator on $\mathcal{H}$ by $I_\mathcal{H}$ and also the standard orthonormal basis of ℓ^2 by $\{\delta_i\}_{i\in I}$.

2. A-duals, approximate A-duals and pseudo A-duals

In order to apply the dual frame expansion (3) we need to find a dual frame. It is usually complicated or impossible to calculate a dual frame explicitly [6, 10, 13, 19]. In the sequel, we introduce a new concept of duality by a left invertible operator and characterize corresponding dual frames.

Definition 2.1. *Let $\Phi = \{\phi_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}$ and $A \in \mathcal{B}^l(\mathcal{H})$. A Bessel sequence $\Psi = \{\psi_i\}_{i\in I}$ is called an A-dual of Φ if $T_\Psi T_\Phi^* A = I_\mathcal{H}$.*

Every frame $\{\phi_i\}_{i\in I}$ has at least one A-dual of the form $\widetilde{\Phi} = \{A^{\ddagger}S_{\Phi}^{-1}\phi_i\}_{i\in I}$ where $A^{\ddagger}$ is a left inverse of A. It is called the canonical A-dual. We begin with the relationship between duals and A-duals.

Lemma 2.1. *Let Φ be a frame and $A \in \mathcal{B}^l(\mathcal{H})$. Then*

(1) $S_{\Phi}^{-1}\Phi$ *is an A-dual of* $(A^{\ddagger})^*\Phi$.
(2) *Every A-dual of Φ is a dual frame for $A^*\Phi$.*
(3) Φ *and* Ψ *are dual frames if and only if Ψ is an A-dual of* $(A^{\ddagger})^*\Phi$.

From now on, we fix the bounded operator $A \in \mathcal{B}^l(\mathcal{H})$ with a left inverse $A^{\ddagger}$.

Proposition 2.1. *Let $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be Bessel sequences and $f, g \in \mathcal{H}$. Then the following are equivalent:*

(1) $f = \sum_{i\in I} \langle Af, \phi_i\rangle \psi_i$.
(2) $f = \sum_{i\in I} \langle f, \psi_i\rangle A^*\phi_i$.
(3) $\langle f, g\rangle = \sum_{i\in I} \langle Af, \phi_i\rangle \langle \psi_i, g\rangle$.

In case the equivalent conditions are satisfied it follows that Ψ is a frame.

Proof. 1. means that $T_{\Psi}T_{\Phi}^*A = I$; this is equivalent to $A^*T_{\Phi}T_{\Psi}^* = I$ which is identical to the statement in 2. It is clear that 1. implies 3. For the converse note that $\sum_{i\in I} \langle Af, \phi_i\rangle \psi_i$, for each $f \in \mathcal{H}$, defines an element of $\mathcal{H}$ because Φ and Ψ are Bessel sequences. Now, 3. shows that

$$\left\langle f - \sum_{i\in I} \langle Af, \phi_i\rangle \psi_i, g \right\rangle = 0, \qquad (g \in \mathcal{H}),$$

and 2. follows. If the conditions are satisfied for Bessel sequences Φ and Ψ, then

$$\begin{aligned} \|f\|^4 &= \left| \sum_{i\in I} \langle Af, \phi_i\rangle \langle \psi_i, f\rangle \right|^2 \\ &\leq \sum_{i\in I} |\langle Af, \phi_i\rangle|^2 \sum_{i\in I} |\langle f, \psi_i\rangle|^2 \\ &\leq C\|A\|^2 \|f\|^2 \sum_{i\in I} |\langle f, \psi_i\rangle|^2, \end{aligned}$$

where C is an upper bound of Φ. Hence, Ψ is a frame. □

Proposition 2.2. *Let $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be Bessel sequences. Then the following statements hold.*

(1) *For any pair of Riesz bases there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that every two Riesz bases are A-dual frames.*
(2) *Let $\Psi = \{\psi_i\}_{i\in I}$ be an A-dual of a Riesz basis $\Phi = \{\phi_i\}_{i\in I}$. Then Ψ is also a Riesz basis if and only if A is invertible.*

Proof. 1. Let $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be two Riesz bases for $\mathcal{H}$. There exists an orthonormal basis $\{e_i\}_{i\in I}$ and bounded invertible operators A_1 and A_2 on $\mathcal{H}$ such that $\phi_i = A_1e_i$ and $\psi_i = A_2e_i$. Consider $A = (A_2A_1^*)^{-1}$. Then $A_2A_1^*A = I$ and for all $f \in \mathcal{H}$ we have

$$\begin{aligned} T_{\Psi}T_{\Phi}^*f &= \sum_{i\in I} \langle Af, \phi_i\rangle \psi_i \\ &= \sum_{i\in I} \langle Af, A_1e_i\rangle A_2e_i \\ &= A_2A_1^*Af = f. \end{aligned}$$

2. The proof is straightforward. □

The following results describe all A-duals of a given frame.

Proposition 2.3. *Let $\Phi = \{\phi_i\}_{i\in I}$ be a frame with bounds C and D. Then $\Psi = \{\psi_i\}_{i\in I}$ is an A-dual frame for Φ if and only if $\psi_i = A^{\ddagger} S_\Phi^{-1}\phi_i + \Theta^*\delta_i$ for some $\Theta \in \mathcal{B}\left(\mathcal{H}, \ell^2\right)$ where $A^*T_\Phi\Theta = 0$.*

Proof. Let $\psi_i = A^{\ddagger} S_\Phi^{-1}\phi_i + \Theta^*\delta_i$ where $\Theta \in \mathcal{B}\left(\mathcal{H}, \ell^2\right)$ and $A^*T_\Phi\Theta = 0$. Then Ψ is a Bessel mapping with the Bessel bound $\left(\|A^{\ddagger}\|\sqrt{C^{-1}} + \|\Theta\|\right)^2$. Also,

$$\begin{aligned} T_\Psi T_\Phi^* A &= \sum_{i\in I} \langle Af, \phi_i\rangle A^{\ddagger} S_\Phi^{-1}\phi_i + \sum_{i\in I} \langle Af, \phi_i\rangle \Theta^*\delta_i \\ &= A^{\ddagger} Af + \Theta^* T_\Phi^* Af = f. \end{aligned}$$

Conversely, suppose Ψ is an A-dual of Φ. Putting $\Theta = T_\Psi^* - T_\Phi^* S_\Phi^{-1}\left(A^{\ddagger}\right)^*$. Then

$$A^*T_\Phi\Theta = A^*T_\Phi T_\Psi^* - A^*T_\Phi T_\Phi^* S_\Phi^{-1}\left(A^{\ddagger}\right)^* = 0.$$

□

Remark 2.1. *Notice that Proposition 2.3 characterizes A-dual frames corresponding to a fixed left invertible operator A. On the other hand, the question is how to characterize the left invertible operator $A \in \mathcal{B}^l(\mathcal{H})$ such that $T_\Psi T_\Phi^* A = I$, for Bessel mappings $\Psi, \Phi \in \mathcal{H}$. In the following, we give some operators $A \in \mathcal{B}^l(\mathcal{H})$ which for a fixed $\Psi \in \mathcal{H}$ satisfy the A-duality condition $T_\Psi T_\Phi^* A = I$.*

1. *If A is invertible, then $A = \left(T_\Psi T_\Phi^*\right)^{-1}$.*
2. *If Ψ is Riesz basis, then Φ is also a Riesz basis and $A = \left(T_\Phi^*\right)^{-1} T_\Psi^{-1}$.*
3. *If Φ is Riesz basis, then $A = \left(T_\Phi^*\right)^{-1} V^*$ for some $V \in \mathcal{B}\left(\ell^2, \mathcal{H}\right)$ by $VT_\Psi^* = I$.*
4. *If Φ is Riesz basis, then $A = \left(T_\Phi^*\right)^{-1} T_\Psi^* S_\Psi^{-1} + \left(T_\Phi^*\right)^{-1}\Theta$ for some $\Theta \in \mathcal{B}\left(\mathcal{H}, \ell^2\right)$ by $T_\Psi\Theta = 0$.*

Theorem 2.1. *Let $\Phi = \{\phi_i\}_{i\in I}$ be a frame. The A-dual frames of Φ are precisely the family $\{V\delta_i\}_{i\in I}$ where $V : \ell^2(\mathbb{N}) \to \mathcal{H}$ is a bounded operator and $VT_\Phi^* A = I$.*

Proof. Let $\Psi = \{V\delta_i\}_{i\in I}$ and $VT_\Phi^* A = I$. Then V^* is bounded below. Hence, Ψ is actually a frame and an A-dual of Φ. In fact,

$$\begin{aligned} T_\Psi T_\Phi^* A &= \sum_{i\in I} \langle Af, \phi_i\rangle \psi_i \\ &= \sum_{i\in I} \langle Af, \phi_i\rangle V\delta_i \\ &= V\sum_{i\in I} \langle Af, \phi_i\rangle \delta_i \\ &= VT_\Phi^* A = I. \end{aligned}$$

Conversely, suppose Ψ is an A-dual. Take $V = T_\Psi$. Then

$$VT_\Phi^* A = T_\Psi T_\Phi^* A = I.$$

□

Here we try to state our results for continuous frames. From now, we assume that $L^2(X)$ admits a Riesz type frame R with the frame bounds C_R and D_R, see [20]. It plays the role of the orthonormal basis of ℓ^2 in discrete case [6].

Definition 2.2. *Let F be a continuous frame for Hilbert space $\mathcal{H}$ and $A \in \mathcal{B}^{\ell}(\mathcal{H})$. A continuous frame G is called an A-dual frame of F, if for every $f \in \mathcal{H}$ we have*

$$f = \int_X \langle Af, F(x)\rangle G(x)\, d\nu(x). \tag{5}$$

From the definition it follows that there exists $A \in \mathcal{B}^{l}(\mathcal{H})$ such that every frame F is an A-dual frame of itself. Recall that a continuous frame is called Riesz type if it has only one dual. Obviously, F is Riesz type frame if its A-dual is Riesz type. Moreover, the A-duality preserves the Riesz type property if and only if A is invertible.

It is worthwhile to mention that the A-duality (5) can rewrite as the operator form

$$T_G T_F^* A = I.$$

Hence, the results proven for discrete frames are also valid for the other type of frames such as continuous frames. For example, we can state Proposition 2.3 and Proposition 2.1 for continuous frames as following.

Proposition 2.4. *Let F be a continuous frame for $\mathcal{H}$ and $A \in \mathcal{B}^{l}(\mathcal{H})$. Then A-dual frames of F are precisely the mappings*

$$G = A^{\ddagger} S_F^{-1} F + \Lambda^* R$$

where $\Lambda \in \mathcal{B}\left(\mathcal{H}, L^2(X)\right)$ and $A^* T_F T_R^* \Lambda = 0$.

Proof. Let $G = A^{\ddagger} S_F^{-1} F + \Lambda^* R$ where $T_F T_R^* \Lambda = 0$. Then G is a Bessel mapping with the upper bound $\left(\|A^{\ddagger}\|\sqrt{D_F} C_F^{-1} + \|\Lambda\|\sqrt{D_R}\right)^2$ which D_R is the upper bound of R and C_F is a lower frame bound of F. Also,

$$T_G T_F^* A = A^{\ddagger} S_F^{-1} S_F A + \Lambda^* T_R T_F^* A = A^{\ddagger} A = I.$$

Conversely, if G is an A-dual of F, then the operator $\Lambda : \mathcal{H} \to L^2(X)$ defined by

$$\Lambda = (T_R^*)^{-1} \left(T_G^* - T_F^* S_F^{-1} \left(A^{\ddagger}\right)^*\right)$$

is a bounded operator. Moreover,

$$A^* T_F T_R^* \Lambda = A^* T_F T_R^* (T_R^*)^{-1} \left(T_G^* - T_F^* S_F^{-1} \left(A^{\ddagger}\right)^*\right) = 0.$$

□

The proof of the following result is similar to the proof of Proposition 2.1, hence it left to the reader.

Proposition 2.5. *Let F be a continuous frame for $\mathcal{H}$ and $A \in \mathcal{B}^{l}(\mathcal{H})$. The A-dual frames for F are precisely the family VR where $V : L^2(X) \to \mathcal{H}$ is bounded operator such that $V T_R T_F^* A = I$.*

The relationship between optimal frame bounds of a frame and its duals is considered by many authors [3, 4]. To obtain these results for A-duals assume that F is a continuous frame with the optimal bounds C_F^{op} and D_F^{op}, respectively. Applying (5) for an A-duals G with the optimal bounds C_G^{op} and D_G^{op}, it follows that

$$\begin{aligned} \|f\|^4 &= |\langle f, f\rangle|^2 \\ &\leq |\langle T_G T_F^* A f, f\rangle|^2 \\ &\leq D_F^{op} \|A\|^2 \|T_G^* f\|^2 \|f\|^2. \end{aligned}$$

for all $f \in \mathcal{H}$. This means that $C_G^{op} \geq \frac{1}{D_F^{op}\|A\|^2}$. Similarly, $\frac{1}{D_G^{op}\|A\|^2} \leq C_F^{op}$.

Two Bessel sequences F and G are said to be approximate A-dual frame if $\|I - T_G T_F^* A\| < 1$, and pseudo A-dual if $T_G T_F^* A$ is invertible [12]. We summarize the relationship between A-duals, approximate A-duals and pseudo A-duals in the following result, which follows immediately from the definition.

Proposition 2.6. *Let F be a continuous frame and G a Bessel mapping. The following assertions hold:*

(1) *Every A-dual is a pseudo A-dual.*
(2) *Every pseudo A-dual is a frame.*
(3) *If G is a pseudo A-dual of F, then $(T_G T_F^* A)^{-1} G$ is an A-dual of G.*
(4) *If G is an approximate A-dual of F, then $(T_G T_F^* A)^{-1} G$ is an A-dual of G.*
(5) *If G is a pseudo A-dual of F, and $W \in \mathcal{B}(\mathcal{H})$ an injective operator, then F and WG are pseudo dual frames.*

A characterization of approximate A-duals and pseudo A-duals is given in the next results.

Proposition 2.7. *Let F and G be continuous frames in $\mathcal{H}$. Then the following are equivalent:*

(1) *G is an approximate A-dual for F.*
(2) *$G = VR$ where $V \in \mathcal{B}\left(L^2(X), \mathcal{H}\right)$ and $\|I_{\mathcal{H}} - V T_R T_F^* A\| < 1$.*
(3) *$G = A^{\ddagger} S_F^{-1} F + K$ where K is a Bessel mapping and $\|T_K T_F^* A\| < 1$.*
(4) *$G = A^{\ddagger} S_F^{-1} F + \Lambda^* R$ where $\Lambda \in \mathcal{B}\left(\mathcal{H}, L^2(X)\right)$ and $\|A^* T_F T_R^* \Lambda\| < 1$.*

Proof. 1. $\Leftrightarrow$ 2. It is evident.
1. $\Leftrightarrow$ 3. Suppose G is an approximate A-dual for F. Denote $K = G - A^{\ddagger} S_F^{-1} F$. Then

$$\|T_K T_F^* A\| = \|I_{\mathcal{H}} - T_G T_F^* A\| < 1.$$

Conversely, if K is a Bessel mapping and $G = A^{\ddagger} S_F^{-1} F + K$ where $\|T_K T_F^* A\| < 1$, then

$$\|I_{\mathcal{H}} - T_G T_F^* A\| = \|I_{\mathcal{H}} - I_{\mathcal{H}} + T_K T_F^* A\| < 1.$$

1. $\Leftrightarrow$ 4. If G is an approximate A-dual for F, then

$$\Lambda = (T_R^*)^{-1} \left(T_G^* - T_F^* S_F^{-1} \left(A^{\ddagger}\right)^*\right).$$

Hence,

$$\|A^* T_F T_R^* \Lambda\| = \left\|A^* T_F T_G^* - A^* T_F T_F^* S_F^{-1} (A^{\ddagger})^*\right\| < 1.$$

The converse follows immediately. $\square$

In the next theorem by using an A-dual (approximate A-dual) of a given frame, we will obtain more A-duals (approximate A-duals).

Theorem 2.2. *Let F be a continuous frame for $\mathcal{H}$. The following assertions hold.*

(1) *A Bessel mapping G is an A-dual of F if and only if*

$$G = A^{\ddagger} S_F^{-1} F + S_F A K - F \tag{6}$$

where K is an A-dual of F.

(2) *If $C_F > \|A^{\ddagger}\|$ and $D_F < \|A\|^{-1}$, then 1. holds for approximate A-duals.*

Proof. 1. Suppose K is an A-dual of F and $G = A^{\ddagger} S_F^{-1} F + S_F A K - F$, then G is an A-dual of F. Indeed

$$T_G T_F^* A = A^{\ddagger} S_F^{-1} T_F T_F^* A + S_F A T_K T_F^* A - T_F T_F^* A = I.$$

Conversely, let G be an A-dual of F. Define $K : X \to \mathcal{H}$ by

$$\langle K(x), A^* f\rangle = \left\langle S_F^{-1} \left(G(x) + F(x) - A^{\ddagger} S_F^{-1} F(x)\right), f\right\rangle, \quad (x \in X, f \in \mathcal{H}) \tag{7}$$

Obviously, K is well-defined since A^* is onto and

$$AK = S_F^{-1}\left(G + F - A^{\ddagger} S_F^{-1} F\right). \tag{8}$$

Hence, $G = A^{\ddagger} S_F^{-1} F + S_F AK - F$ and K is a Bessel mapping. Moreover,

$$\begin{aligned} T_K T_F^* A &= A^{\ddagger} S_F^{-1} T_G T_F^* A + A^{\ddagger} S_F^{-1} T_F T_F^* A - A^{\ddagger} S_F^{-1} A^{\ddagger} S_F^{-1} T_F T_F^* A \\ &= A^{\ddagger} S_F^{-1} + I_{\mathcal{H}} - A^{\ddagger} S_F^{-1} = I_{\mathcal{H}}. \end{aligned}$$

2. Let G be an approximate A-dual of F. Define $K : X \to \mathcal{H}$ by (7), consequently, by using (8) we obtain

$$\begin{aligned} \|I - T_K T_F^* A\| &= \|I - A^{\ddagger} S_F^{-1} T_G T_F^* A - A^{\ddagger} S_F^{-1} T_F T_F^* A + A^{\ddagger} S_F^{-1} A^{\ddagger} S_F^{-1} S_F A\| \\ &\leq \|A^{\ddagger} S_F^{-1}\| \|I_{\mathcal{H}} - T_G T_F^* A\| \\ &\leq \|A^{\ddagger}\| C_F^{-1} < 1. \end{aligned}$$

Conversely, assume that K is an approximate A-dual of F and G is defined by (6), then

$$\begin{aligned} \|I - T_G T_F^* A\| &= \|I - A^{\ddagger} S_F^{-1} S_F A - S_F A T_K T_F^* A - S_F A\| \\ &\leq \|S_F A\| \|I_{\mathcal{H}} - T_K T_F^* A\| \\ &\leq D_F \|A\| < 1. \end{aligned}$$

□

3. **Perturbation of A-duals**

Stability and perturbation of frames have been investigated in [11, 22]. In [14], the perturbation theory for A-dual frames is considered when A is invertible. In the sequel, we investigate perturbation of approximate A-duals. That is, if G is an A-dual of a continuous frame F, and $\widetilde{F}$ is a Bessel mapping in $\mathcal{H}$ which is in some sense closed to F, does it follows that G is an approximate A-dual of $\widetilde{F}$? The following results make some conditions under which G is an approximate A-dual of $\widetilde{F}$. For example, if

$$\int_X \left|\left\langle f, F(x) - \widetilde{F}(x)\right\rangle\right|^2 d\nu(x) \leq \frac{\|f\|^2}{\|A\|\sqrt{D_G}}, \quad (f \in \mathcal{H}).$$

Then

$$\left\|\left(T_F^* - T_{\widetilde{F}}^*\right) f\right\|^2 = \int_X \left|\left\langle f, F(x) - \widetilde{F}(x)\right\rangle\right|^2 d\nu(x) \leq \frac{\|f\|^2}{\|A\|\sqrt{D_G}},$$

and therefore,

$$\begin{aligned} \left\|I_{\mathcal{H}} - T_G T_{\widetilde{F}}^* A\right\| &= \left\|T_G T_F^* A - T_G T_{\widetilde{F}}^* A\right\| \\ &\leq \|T_G\| \left\|T_F^* - T_{\widetilde{F}}^*\right\| \|A\| < 1. \end{aligned}$$

Hence, $\left(T_G T_{\widetilde{F}}^* A\right)^{-1} G$ is an A-dual of $\widetilde{F}$.

Proposition 3.1. *Let G be an A-dual of a continuous frame F and $\widetilde{G}$ be a Bessel mapping such that that*

$$\left\|\int_X \phi(x)\left(G(x) - \widetilde{G}(x)\right) d\nu(x)\right\| \leq \lambda \left\|\int_X \phi(x) G(x) d\nu(x)\right\| + \gamma \left(\int_X |\phi(x)|^2 d\nu(x)\right)^{1/2} \tag{9}$$

for all $\phi \in L^2(X)$ and for some positive numbers λ, γ with $\lambda + \gamma\sqrt{D_G}\|A\| < 1$. Then $\widetilde{G}$ is an approximate A-dual of F, In particular, $\left(T_{\widetilde{G}} T_F^* A\right)^{-1} \widetilde{G}$ is an A-dual of F.

Proof. Using (9) for all $f \in \mathcal{H}$ we obtain

$$\begin{aligned}\left\|f - T_{\widetilde{G}}T_F^*Af\right\| &= \left\|T_GT_F^*Af - T_{\widetilde{G}}T_F^*Af\right\| \\ &= \left\|\int_X \langle Af, F(x)\rangle \left(G(x) - \widetilde{G}(x)\right) d\nu(x)\right\| \\ &\leq \lambda \left\|\int_X \langle Af, F(x)\rangle G(x) d\nu(x)\right\| + \gamma \left(\int_X |\langle Af, F(x)\rangle|^2 d\nu(x)\right)^{1/2} \\ &\leq \left(\lambda + \gamma\sqrt{D_G}\|A\|\right)\|f\| < \|f\|.\end{aligned}$$

□

In the following theorem, from an approximate A-dual which is closed to an A-dual G, we can construct an A-dual which is closed to G.

Theorem 3.1. *Let G be an A-dual of a continuous frame F.*

(1) *If $\widetilde{G}$ is a continuous frame such that*

$$\left\|T_G - T_{\widetilde{G}}\right\| < \frac{1}{\sqrt{D_F}\|A\|} \tag{10}$$

then $\widetilde{G}$ is an approximate A-dual of F.

(2) *If (10) holds and $C_F > \|A^\ddagger\|$, then there exist an A-dual $\widetilde{\widetilde{G}}$ of F and a constant $M > 0$ such that*

$$\left\|T_G - T_{\widetilde{\widetilde{G}}}\right\| \leq M \left\|T_G - T_{\widetilde{G}}\right\|.$$

Proof. 1. Suppose G is an A-dual of F. Using Theorem 2.2 there exists an A-dual K of F such that

$$G = A^\ddagger S_F^{-1}F + S_FAK - F.$$

Define $\widetilde{K} : X \to \mathcal{H}$ by

$$\left\langle \widetilde{K}(x), A^*f \right\rangle = \left\langle AK(x) - S_F^{-1}\left(G(x) - \widetilde{G}(x)\right), f\right\rangle, \quad (x \in X, f \in \mathcal{H}).$$

Obviously, $\widetilde{K}$ is well-defined since A^* is onto and

$$A\widetilde{K} = AK - S_F^{-1}\left(G - \widetilde{G}\right).$$

Hence, $G - \widetilde{G} = S_FA\left(K - \widetilde{K}\right)$, and therefore

$$\left\|T_K - T_{\widetilde{K}}\right\| \leq \left\|A^\ddagger\right\| \left\|S_F^{-1}\right\| \left\|T_G - T_{\widetilde{G}}\right\|. \tag{11}$$

By (10) we have

$$\left\|I - T_{\widetilde{G}}T_F^*A\right\| = \left\|T_GT_F^*A - T_{\widetilde{G}}T_F^*A\right\| \leq \left\|T_G - T_{\widetilde{G}}\right\| \|T_F^*\| \|A\| < 1,$$

Thus, $\widetilde{G}$ is an approximate A-dual of F.

2. According to the above computations and by using (10) and (11) we obtain

$$\begin{aligned}\|I - T_{\widetilde{K}}T_F^*A\| &\leq \|T_KT_F^*A - T_{\widetilde{K}}T_F^*A\| \\ &\leq \|T_K - T_{\widetilde{K}}\|\|T_F^*\|\|A\| \\ &\leq \|T_G - T_{\widetilde{G}}\|\|S_F^{-1}\|\|A^\ddagger\|\|T_F^*\|\|A\| \\ &\leq \|A^\ddagger\|C_F^{-1} < 1.\end{aligned}$$

So, $T_{\widetilde{K}}T_F^*A$ is invertible. Using Proposition 2.2 of [7], it follows that

$$\|\left(T_{\widetilde{K}}T_F^*A\right)^{-1}\| \leq \frac{1}{1 - \|I - T_{\widetilde{K}}T_F^*A\|} \leq \frac{1}{1 - \|A^\ddagger\|C_F^{-1}}.$$

In particular,

$$\begin{aligned}\|I-\left(T_{\widetilde{K}}T_F^*A\right)^{-1}\| &\leq \|\left(T_{\widetilde{K}}T_F^*A\right)^{-1}\|\|I-T_{\widetilde{K}}T_F^*A\|\\ &\leq \frac{1}{1-\|A^\ddagger\|C_F^{-1}}\|T_G-T_{\widetilde{G}}\|\|S_F^{-1}\|\|A^\ddagger\|\|T_F^*\|\|A\|\\ &\leq \frac{\|A\|\|A^\ddagger\|D_FC_F^{-1}}{1-\|A^\ddagger\|C_F^{-1}}\|T_G-T_{\widetilde{G}}\|.\end{aligned}$$

Denote $\widetilde{\widetilde{G}}=A^\ddagger S_F^{-1}F+S_FA\widetilde{\widetilde{K}}-F$, where $\widetilde{\widetilde{K}}=\left(T_{\widetilde{K}}T_F^*A\right)^{-1}\widetilde{K}$. Then the Bessel mapping $\widetilde{\widetilde{G}}$ is an A-dual of F. In fact,

$$\begin{aligned}T_{\widetilde{\widetilde{G}}}T_F^*A &= \left(A^\ddagger S_F^{-1}T_F+S_FA\left(T_{\widetilde{K}}T_F^*A\right)^{-1}T_{\widetilde{K}}-T_F\right)T_F^*A\\ &= I+S_FA-S_FA=I.\end{aligned}$$

$$\begin{aligned}\|T_{\widetilde{K}}\| &= \|T_K-\left(T_K-T_{\widetilde{K}}\right)\|\\ &= \|T_K-A^\ddagger S_F^{-1}\left(T_G-T_{\widetilde{G}}\right)\|\\ &\leq \|T_K\|+\|A^\ddagger\|\|S_F^{-1}\|\|T_G-T_{\widetilde{G}}\|\\ &\leq \|A^\ddagger S_F^{-1}\left(T_G+T_F+A^\ddagger S_F^{-1}T_F\right)\|+\|A^\ddagger\|\|S_F^{-1}\|\|T_G-T_{\widetilde{G}}\|\leq\alpha,\end{aligned}$$

where $\alpha=\|A^\ddagger\|C_F^{-1}\left(\sqrt{D_G}+\sqrt{D_F}+\|A^\ddagger\|C_F^{-1}\sqrt{D_F}+\|T_G-T_{\widetilde{G}}\|\right)$. Therefore,

$$\begin{aligned}\|T_G-T_{\widetilde{\widetilde{G}}}\| &\leq \|S_F\|\|A\|\|T_K-T_{\widetilde{\widetilde{K}}}\|\\ &\leq \|S_F\|\|A\|\left(\|T_K-T_{\widetilde{K}}\|+\|T_{\widetilde{K}}-T_{\widetilde{\widetilde{K}}}\|\right)\\ &\leq \|S_F\|\|A\|\left(\|T_G-T_{\widetilde{G}}\|\|S_F^{-1}\|\|A^\ddagger\|+\|T_{\widetilde{K}}-\left(T_{\widetilde{K}}T_F^*A\right)^{-1}T_{\widetilde{K}}\|\right)\\ &\leq \|S_F\|\|A\|\left(\|T_G-T_{\widetilde{G}}\|\|S_F^{-1}\|\|A^\ddagger\|+\|T_{\widetilde{K}}\|\|I-\left(T_{\widetilde{K}}T_F^*A\right)^{-1}\|\right)\\ &\leq D_FC_F^{-1}\|A\|\|A^\ddagger\|\|T_G-T_{\widetilde{G}}\|+D_F\|A\|\left(\frac{\|A\|\|A^\ddagger\|D_FC_F^{-1}\alpha}{1-\|A^\ddagger\|C_F^{-1}}\|T_G-T_{\widetilde{G}}\|\right)\\ &\leq \left(D_FC_F^{-1}\|A\|\|A^\ddagger\|+\frac{\|A\|^2\|A^\ddagger\|D_F^2C_F^{-1}\alpha}{1-\|A^\ddagger\|C_F^{-1}}\right)\|T_G-T_{\widetilde{G}}\|.\end{aligned}$$

This completes the proof. □

In [5], by observing the difference between an alternate dual and the canonical dual, a stability result for alternate duals of g-frames is introduced. Inspired by this idea and the above results we state the stability of A-duals of continuous frames as follows.

Corollary 3.1. *Let F and $\widetilde{F}$ be two continuous frames and G be a fixed A-dual of F. If $F-\widetilde{F}$ is a Bessel mapping with sufficiently small bound $\epsilon>0$, then there exists a dual $\widetilde{G}$ for $\widetilde{F}$ such that $\widetilde{G}-G$ is also a Bessel mapping.*

4. **Conclusion**

The frame theory has many applications in engineering, image processing and signal processing. In most of these applications, we deals with a dual frame to reconstruct a function or signal.The classical choice in this case is the canonical dual. We introduced the concept of A-duals corresponding to a left invertible operator A. Then we characterized A-duals and obtained more reconstruction formulas. Finally, some stability results of A-duals of continuous frames are stated.

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