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STABILIZER IN RESIDUATED LATTICES

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In this paper we introduce the notion of (right and left) stabilizer in residuated lattices, we state and prove some theorems which determine the relationship between this notion and all types of filters in residuated lattices. After that we construct quotient of residuated lattices via stabilizer and study its properties.

Keywords: Residuated lattices, (Implicative, Positive implicative, Fantastic, Boolean, Obstinate) Filter, Gödel algebra, MV-algebra, (right and left) stabilizer.

1. Introduction

A commutative integral residuated bounded lattice is an algebraic structure $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, *, 1)$ is a commutative monoid and, for all $a, b, c \in A$,

$a \leq b \rightarrow c$ if and only if $a * b \leq c$.

Commutative integral residuated bounded lattices have been studied extensively and include important classes of algebras such as BL-algebras, introduced by Hájek as the algebraic counterpart of Basic Logic [8], and MV-algebras, the algebraic setting for Lukasiewicz propositional logic (we refer to the monograph [5] for a detailed treatment of MV-algebras). In Y. Zhu et al(2009) introduced the notion of implicative(Boolean) filter and fantastic filter of residuated lattice.

Now, in this note we introduce the notions of (left, right) stabilizer of X and (left, right) stabilizer of X with respect to Y, for subsets X and Y of A. We show that right stabilizer of X is a filter but is not true for left and also right stabilizer of X with respect to Y is a filter. Then we study some properties of them.

2. Preliminaries

Definition 2.1. [15] An algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $< 2, 2, 2, 2, 0, 0 >$ is called residuated lattice if satisfies:

(LR_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(LR_2) $(A, *, 1)$ is a commutative monoid,

(LR_3) $*$ and $\rightarrow$ form an adjoint pair i.e, $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in A$.

A residuated lattice, A is called a Gödel algebra if $x^2 = x * x = x$, for all $x \in A$

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and residuated lattice, A is called an MV-algebra if $\neg(\neg x) = x$ or equivalently $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in A$, where $\neg x = x \to 0$.

Lemma 2.2. [8] In each residuated lattice A, the following relations hold for all $x, y, z \in A$:
(1) $x * (x \to y) \leq y$,
(2) $x \leq (y \to (x * y))$,
(3) $x \leq y$ iff $x \to y = 1$,
(4) $x \to (y \to z) = y \to (x \to z)$,
(5) If $x \leq y$, then $y \to z \leq x \to z$ and $z \to x \leq z \to y$,
(6) $y \leq (y \to x) \to x$,
(7) $y \to x \leq (z \to y) \to (z \to x)$,
(8) $x \to y \leq (y \to z) \to (x \to z)$,
(9) $x \leq y$ implies $x * z \leq y * z$,
(10) $1 \to x = x$, $x \leq y \to x$, $x * y \leq x \wedge y$,
(11) $x * \neg x = 0$,
(13) $x * y = 0$ iff $x \leq \neg y$ and $x \leq y$ implies $\neg y \leq \neg x$,
(14) $x \to y \leq (x * z) \to (y * z)$,
(15) $x * (y \to z) \leq y \to (x * z)$,
(16) $(y \to z) * (x \to y) \leq (x \to z)$,
(17) $x \leq \neg\neg x$, $\neg 1 = 0$, $\neg 0 = 1$, $\neg\neg\neg x = \neg x$, $\neg\neg x \leq \neg x \to x$
(18) $\neg\neg(x * y) = \neg\neg x * \neg\neg y$,
(19) $x = \neg\neg x * (\neg\neg x \to x)$,
(20) if $\neg\neg x \leq \neg\neg x \to x$, then $\neg\neg x = x$,
(21) $x \to \neg y = y \to \neg x = \neg\neg x \to \neg y = \neg(x * y)$,
(22) $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$,
(23) $x \to y = y \to x = 1$ iff $x = y$, $x \to 1 = 1$, $0 \to x = 1$,
(24) $x \to y \leq (x * z) \to (y * z)$,
(25) $(a \vee b) \to x = (a \to x) \wedge (b \to x)$,
(26) $(x * y) \to z = x \to (y \to z)$.

For any residuated lattice A, $B(A)$ denotes the Boolean algebra of all complemented elements in $L(A)$ (hence $B(A) = B(L(A))$).

Proposition 2.3. [14] For $e \in A$, the following are equivalent:
(i) $e \in B(A)$,
(ii) $e * e = e$ and $e = \neg\neg e$,
(iii) $e * e = e$ and $\neg e \to e = e$,
(iv) $e \vee \neg e = 1$,
(v) $(e \to x) \to e = e$, for every $x \in A$.

Definition 2.4. [14] A filter of a residuated lattice A is a nonempty subset F of A such that for all $a, b \in A$, we have:

(1) $a, b \in F$ implies $a * b \in F$,

(2) $a \in F$ and $a \leq b$ imply $b \in F$.

An alternative definition for a filter F of a residuated lattice A is the following:

(1) $1 \in F$,

(2) for all x and y in A: if $x, x \to y \in F$, then $y \in F$.

The set of all filters of A is denoted by $\mathbf{F}$.

Definition 2.5. [14] A proper filter M of a residuated lattice A is called maximal (or ultrafilter) if it is not properly contained in any other proper filter of A.

Definition 2.6. [14] Let A be a residuated lattice and F be a filter of A. F is called a prime filter if $x \vee y \in F$ implies $x \in F$ or $y \in F$.
Let A be a residuated lattice and F be a filter of A. F is a prime filter iff $(x \to y) \in F$ or $(y \to x) \in F$, for all $x, y \in A$.

Theorem 2.7. [14] Let F be a filter of a residuated lattice A. Define:

$$x \equiv_F y \ \text{ iff } \ x \to y \in F \ \text{ and } \ y \to x \in F.$$

Then $\equiv_F$ is a congruence relation on A.

The set of all congruence classes is denoted by A/F, i.e., $A/F := \{[x] | x \in A\}$, where $[x] = \{y \in A | x \equiv_F y\}$.

Defines $\bullet, \to, \sqcap, \sqcup$ on A/F, as follows:
$[x] \bullet [y] = [x * y]$,
$[x] \to [y] = [x \to y]$,
$[x] \sqcap [y] = [x \wedge y]$,
$[x] \sqcup [y] = [x \vee y]$,
Therefore $(A/F, \sqcap, \sqcup, \bullet, \to, [1], [0])$ is a residuated lattice which is called quotient residuated lattice with respect to F.

Definition 2.8. [2] A nonempty subset F of A is called:

A Boolean filter of A if F is a filter of A and $x \vee (\neg x) \in F$,

An implicative filter of A if $1 \in F$ and $x \to (y \to z) \in F$ and $x \to y \in F$ imply that $x \to z \in F$,

A positive implicative filter of A if $1 \in F$ and $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$,

A fantastic filter of A if $1 \in F$ and $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$,

An obstinate filter of A if and only if be a proper filter and $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$,

for all $x, y, z \in A$.

3. **Stabilizer in residuated lattice**

From now on $(A, \wedge, \vee, *, \to, 0, 1)$ or simply A is a residuated lattice unless otherwise specified.

Definition 3.1. Let X, Y are non-empty subset of A. We define:

$$X_R^* = \{a \in A : a \to x = x, \forall x \in X\},$$
$$X_L^* = \{a \in A : x \to a = a, \forall x \in X\},$$

X_R^* and X_L^* are called right and left stabilizer of A and denote stabilizer of X by $X^* = X_R^* \cap X_L^*$.
We define stabilizer of X with respect to Y or $(X, Y)^* = (X, Y)_R^* \cap (X, Y)_L^*$ where:

$$(X, Y)_R^* = \{a \in A : (a \to x) \to x \in Y, \forall x \in X\},$$
$$(X, Y)_L^* = \{a \in A : (x \to a) \to a \in Y, \forall x \in X\}.$$

Example 3.2. Let $A = \{0, a, b, c, 1\}$, where $0 < a < b < c < 1$. Define on A the following operations:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a residuated lattice. Then $\{b\}^*_R = \{x \in A : x \to b = b\} = \{1, c\}$ and $\{b\}^*_L = \{x \in A : b \to x = x\} = \{1, 0\}$. Therefore $\{b\}^* = \{1\}$. The other side we have $(\{a\}, \{b\})^*_R = \{x \in A : (x \to a) \to a = b\} = \{b\}$ and $(\{a\}, \{b\})^*_L = \{x \in A : (a \to x) \to x = b\} = \{b\}$. Hence $(\{a\}, \{b\})^* = \{b\}$.

Proposition 3.3. Let A be a residuated lattice and $X, Y \subseteq A$. Then:
(1) If $X \subseteq Y$, then $Y^* \subseteq X^*$,
(2) $A^* = \{1\}$ and $\{1\}^* = A$,
(3) $X^* = \bigcap\{\{x\}^* : x \in X\}$,
(4) If $h : A \to A$ be a homomorphism and $a \in A$, then $h(\{a\}^*) \subseteq \{h(a)\}^*$.

Proof.(1) Let $a \in Y^* = Y^*_R \cap Y^*_L$. Then $a \to x = x$ and $x \to a = a$, for all $x \in Y$. Since $X \subseteq Y$, we have $a \to x = x$ and $x \to a = a$ for all $x \in X$, that is $a \in X^*$.

(2) Let $a \in A^*$, hence $a \to b = b$ and $b \to a = a$, for all $b \in A$. If we suppose $b = a$, then $1 = a \to a = a$. Therefore $a = 1$ and $A^* \subseteq \{1\}$.

On the other hand we can say $1 \to b = b$ and $b \to 1 = 1$, for all $b \in A$. Then $1 \in A^*$. Therefore $A^* = \{1\}$. Let $a \in A$, we can say $a \in \{1\}^*$, since $1 \to a = a$ and $a \to 1 = 1$. Then $A \subseteq \{1\}^*$. On the other hand we have $\{1\}^* \subseteq A$. Therefore $\{1\}^* = A$.

(3)$a \in X^*$ if and only if $a \to x = x$ and $x \to a = a$, for all $x \in X$ if and only if $a \in \{x\}^*$, for all $x \in X$.

(4) Let $a \in A$, $h : A \to A$ be a homomorphism and $y \in h(\{a\}^*)$. Then there exists $x \in \{a\}^*$ such that $y = h(x)$. Hence $x \to a = a$ and $a \to x = x$ and since h is a homomorphism we get $h(x) \to h(a) = h(a)$ and $h(a) \to h(x) = h(x)$, that is $y = h(x) \in \{h(a)\}^*$.□

Theorem 3.4. Let A be a residuated lattice and $X \subseteq A$. Then X^*_R is a filter of A.

Proof. Since $1 \to x = x$, for all $x \in X$, we have $1 \in X^*_R$. Now let $a, a \to b \in X^*_R$, then $a \to x = x$ and $(a \to b) \to x = x$, for all $x \in X$ and so by Lemma 2.2 we get:

$b \to x \le (a \to b) \to (a \to x) = a \to ((a \to b) \to x) = a \to x = x$ and $x \le b \to x$. Then $b \to x = x$, for all $x \in X$.□

The following example we show that not every X^*_L is a filter of A and we will check the relationship between $(X^*_L)^*_L$, X^*_L and $(X^*_R)^*_R$, X^*_R.

Example 3.5. Let $A = \{0, a, b, c, 1\}$. Define on A the following operations:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Easily we can check that $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a residuated lattice. Then $\{b\}^*_L = \{x \in A : b \rightarrow x = x\} = \{a, 1\}$, but $\{a, 1\}$ is not a filter because $a \leq c$ and $a \in \{b\}^*_L$ but $c \notin \{b\}^*_L$.

Now, if $X = \{b\}$, then we see $\{b\}^*_L \neq (\{b\}^*_L)^*_L$ and $\{b\}^*_R \neq (\{b\}^*_R)^*_R$ since $\{b\}^*_L = \{a, 1\}$ and $(\{b\}^*_L)^*_L = \{b, 1\}$ and $\{b\}^*_R = \{a, 1\}$ and $(\{b\}^*_R)^*_R = \{c, b, 1\}$.

Also, in the following example we show that X^*_L is not a filter whenever X is a filter of A.

Example 3.6. Let $A = \{0, a, b, c, 1\}$. Define on A the following operations:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Then we can see that A is a residuated lattice. It is clear that $F = \{1, c\}$ is a filter of A, but $F^*_L = \{1, a, b\}$ is not a filter of A.

Corollary 3.7. Let A be a residuated lattice and $X \subseteq A$. Then $A/X^*_R = \{[b] : b \in A\}$ where $[b] = \{a \in A : (a \rightarrow b) * (b \rightarrow a) \rightarrow x = x, \forall x \in X\}$.

Theorem 3.8. Let F and G be filters of residuated lattice A. Then $(F, G)^*_R$ is a filter of A.

Proof. Let F and G be filter of residuated lattice A. Then $1 \in (F, G)^*_R$, since $(1 \rightarrow x) \rightarrow x = 1 \in G$, for all $x \in F$ and if there exist $a, b \in A$ such that $a \leq b$ and $a \in (F, G)^*_R$, we will have $(a \rightarrow x) \rightarrow x \in G$ and $(a \rightarrow x) \rightarrow x \leq (b \rightarrow x) \rightarrow x$, for all $x \in F$. Hence $(b \rightarrow x) \rightarrow x \in G$, for all $x \in F$. Then $(F, G)^*_R$ is a filter of A.$\square$

In Example 3.5, we can see $F = \{1, a, c\}$ and $G = \{1\}$ are filters of A, but $(F, G)^*_L = \{y \in A : (x \rightarrow y) \rightarrow y \in G, \forall x \in F\} = \{1, b\}$ is not a filter of A.

By the following example we show that the condition F and G be filters of A in Theorem 3.8 is necessary.

Example 3.9. Let $A = \{0, a, b, 1\}$. Define on A the following operation $*$, $\rightarrow$:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Easily we can check that $(A, \wedge, \vee, *, \to, 0, 1)$ is a residuated lattice. Then in A, $\{0\}$ and $\{b, 1\}$ are not filters and also $(\{0\}, \{b, 1\})^*_R = \{0, b, 1\}$ and $(\{b, 1\}, \{0\})^*_R = \emptyset$ are not filters.

Theorem 3.10. Let A be a residuated lattice, F, G filters of A and X, Y, X_i and Y_i for all $i \in I$, subsets of A such that $\cap_{i\in I} X_i \neq \emptyset$ and $\cap_{i\in I} Y_i \neq \emptyset$. Then

(1) If $(X, Y)^*_L = A$ or $(X, Y)^*_R = A$, then $X \subseteq Y$,
(2) $(F, Y)^*_L = (F, Y)^*_R = A$ iff $F \subseteq Y$,
(3) $(F, F)^* = A$,
(4) $X^*_R \subseteq (X, G)^*_R$(hence $X^*_R \subseteq (X, X^*_R)^*_R$) and $X^*_L \subseteq (X, G)^*_L$,
(5) If $X_i \subseteq X_j$ and $Y_i \subseteq Y_j$, then $(Xj, Yi)^*_R \subseteq (Xi, Yj)^*_R$ and $(Xj, Yi)^*_L \subseteq (Xi, Yj)^*_L$. Hence $(Xj, Yi)^* \subseteq (Xi, Yj)^*$,
(6) $(X, \{1\})^*_R = X^*_R$ and $(X, \{1\})^*_L = X^*_L$. Hence $(X, \{1\})^* = X^*$,
(7) $(\{1\}, G)^*_R = A$ and $(\{1\}, G)^*_L = A$. Hence $(\{1\}, G)^* = A$,
(8) $(X, \cap Yi)^*_R \subseteq \cap(X, Yi)^*_R$ and $(X, \cap Yi)^*_L \subseteq \cap(X, Yi)^*_L$. Hence $(X, \cap Yi)^* \subseteq \cap(X, Yi)^*$,
(9) If $x \in \cap\{Y_i\}^*_R$, then $x \in \{\cap Y_i\}^*_R$ and $\cap\{Y_i\}^*_L \subseteq \{\cap Y_i\}^*_L$,
(10) $\bigcap(X_i, Y_i)^*_R = (\cap X_i, \cap Y_i)^*_R$, for all $i \in I$,
(11) $\bigcup(X_i, Y_i)^*_R \subseteq (\cap X_i, \cup Y_i)^*_R$, for all $i \in I$,
(12) $(\cup X_i, \cup Y_i)^*_R \subseteq (\cap X_i, \cup Y_i)^*_R$, for all $i \in I$,
(13) Let $a \in X^*_R$. Then $a^n \in X^*_R$, for all $n \in N$,
(14) $F \neq \cap\{a\}^*_R$ and $F \neq \cap\{a\}^*_L$, where $a \in F$,
(15) $F \subseteq \cup\{a\}^*_R = A$ and $F \subseteq \cup\{a\}^*_L = A$, where $a \in F$.

Proof. (1) Let $(X, Y)^*_R = A$. Since $0 \in A$ we have $0 \in (X, Y)^*_R$ and so $x = (0 \to x) \to x \in Y$, for all $x \in X$, that is $X \subseteq Y$. Let $(X, Y)^*_L = A$ and $x \in X \subseteq A$. Thus $x \in (X, Y)^*_L$ and $(x \to x) \to x \in Y$. Therefore $x \in Y$.
(2) Let $F \subseteq Y$ and $a \in A$, $x \in F$ and F be a filter. Thus $x \in Y$. Since $x \leq (a \to x) \to x$, we get that $(a \to x) \to x \in Y$. Thus $a \in (F, Y)^*_R$. Similarly $a \in (F, Y)^*_L$. The converse is clear by (1).
(3) It is clear by (2), since F is a filter and $F \subseteq F$.
(4) Let $a \in X^*_L$. Then $x \to a = a$, for all $x \in X$. Since G is a filter $(x \to a) \to a = a \to a = 1 \in G$, that is $a \in (X, G)^*_L$. Similarly $X^*_R \subseteq (X, G)^*_R$ become apparent.
(5) Let $x \in (X_j, Y_i)^*_L$. Then $(a \to) x \to x \in Y_i$, for all $a \in X_j$ and so $(a \to x) \to x \in Y_j$, for all $x \in X_i$. Hence $a \in (X_i, Y_j)^*_L$. We can Similarly prove $(X_j, Y_i)^*_R \subseteq (X_i, Y_j)^*_R$. Hence $(X_j, Y_i)^* \subseteq (X_i, Y_j)^*$,
(6) Let $a \in (X, \{1\})^*_L$. Then $(x \to a) \to a \in \{1\}$, for all $x \in X$, that is $x \to a \leq a$. Hence $x \to a = a$, for all $x \in X$ and so $a \in X^*_L$. Hence $(X, \{1\})^*_L \subseteq X^*_L$.
Conversely, let $a \in X^*_L$. Then $x \to a = a$, for all $x \in X$ and so $(x \to a) \to a = 1$, for all $x \in X$. Hence $(X, \{1\})^*_L \subseteq X^*_L$. We can Similarly prove $(X, \{1\})^*_R = X^*_R$. Hence $(X, \{1\})^* = X^*$.
(7) Let $a \in A$. Since G is a filter, we have $(1 \to a) \to a = 1 \in G$, hence $a \in (\{1\}, G)^*_L$

and $A = (\{1\}, G)^*_L$. We have $(\{1\}, G)^*_R = A$. Therefore $(\{1\}, G)^* = A$.
(8) Let $a \in (X, \cap Y_i)^*_L$. Then $(x \to a) \to a \in \cap Y_i$, for all $x \in X$. Hence $x \in \cap(X, Y_i)^*_L$. Hence $(X, \cap Y_i)^* \subseteq \cap(X, Y_i)^*$.
(9) Let $x \in \cap\{Y_i\}^*_R$ where $i \in I$. Then $x \in \{Y_i\}^*_R$, for all $i \in I$, then $x \to y_i = y_i$, for all $y_i \in Y_i$, hence $x \to t = t$, for all $t \in \cap Y_i$. Therefore we conclude that $x \in \{\cap Y_i\}^*_R$.
(10) $x \in \bigcap(X_i, Y_i)^*_R$, where $i \in I$ iff $x \in (X_i, Y_i)^*_R$, for all $i \in I$ iff $(x \to y) \to y \in Y_i$, for all $y \in X_i$ and $i \in I$ iff $(x \to y) \to y \in \cap Y_i$, for all $y \in \cap X_i$ iff $x \in (\cap X_i, \cap Y_i)^*_R$.
(11) Let $x \in \bigcup(X_i, Y_i)^*_R$. Then there exists $i \in I$ such that $x \in (X_i, Y_i)^*_R$. Hence there exists $i \in I$ such that for all $y \in X_i$, $(x \to y) \to y \in Y_i$. Therefore we have $(x \to y) \to y \in \cup Y_i$, for all $y \in \cap X_i$ and $x \in (\cap X_i, \cup Y_i)^*_R$, for all $i \in I$.
(12) Let $y \in (\cup X_i, \cup Y_i)^*_R$. Then $\forall x \in \cup X_i$, $(y \to x) \to x \in \cup Y_i$. Hence $\forall x \in \cap X_i$, $(y \to x) \to x \in \cup Y_i$.
(13) Let $a \in X^*_R$. Then $a \to x = x$, for all $x \in A$, hence $a^2 \to x = a \to (a \to x) = a \to x = x$. Therefore $a^2 \in X^*_R$, since $a^2 \to x = x$. Now, let $a^{n-1} \in X^*_R$. Then $a^{n-1} \to x = x$, for all $x \in A$, hence $a^n \to x = a \to (a^{n-1} \to x) = a \to x = x$. Therefore $a^n \in X^*_R$. By induction we have $a^n \in X^*_R$, for all $n \in N$.
(14) Let $a \in F$. Then $a \to a = 1 \neq a$, hence $a \notin \{a\}^*_R$. Therefore $a \notin \cap\{a\}^*_R$, where $a \in F$. On other hand if $x \in \cap\{a\}^*_R$, where $a \in F$ and $x \in F$, we can say $x \to a = a$, for all $a \in F$. If $x \in F$, we have $a \in F$ such that $x = a$. Hence $1 = x \to a = a$ and $F \neq \{1\}$, which is contradiction. We clearly prove $F \neq \cap\{a\}^*_L$, where $a \in F$.
(15) Let F be a filter of A. $\cup\{a\}^*_R = \cup\{a\}^*_L = A$, where $a \in F$, since $1 \in F$ and $1 \to x = x, x \to 1 = 1$, for all $x \in A$. Then $\cup\{a\}^*_R = \cup\{a\}^*_L = A$. □

Theorem 3.11. (1) Let $X \subseteq A$ and $0 \in X^*_L$. Then $D_s(X) = \{x \in X : \neg x = 0\} = X$,
(2) $D_s(A) = (\{0\}, \{1\})^*_R = \{0\}^*_R$,
Proof. (1) Let $X \subseteq A$ and $0 \in X^*_L$. Then $x \to 0 = \neg x = 0$, for all $x \in X$. Hence $D_s(X) = X$.

(2) $x \in \{0\}^*_R$ iff $\neg x = x \to 0 = 0$ iff $x \in D_s(A)$ iff $\neg\neg x = 1$ iff $(x \to 0) \to 0 \in \{1\}$ iff $x \in (\{0\}, \{1\})^*_R$.

In the following theorem we give relationship between stabilizer and Boolean center

Theorem 3.12. (1) Let $\{\neg a\}^*_L \subseteq B(A)$. Then $\neg a \in \{\{\neg a\}^*_L\}^*_R$, ,
(2) Let $\{a\}^*_R \subseteq B(A)$. Then $a \in \{\neg x : x \in \{a\}^*_R\}^*_L$,
(3) $X^*_R \subseteq B(X)$ iff $X^*_R \subseteq X^*_L$ and $X^* = X^*_R$,
(4) Let $B(A) = \{0, 1\}$, $\{1\} \neq X$. Then $X^* = \{1\}$,
(5) Let $a \in B(A)$, then $\neg a \in \{\neg a\}^*_R$.

Proof. (1) Let $x \in \{\neg a\}^*_L$. Then $\neg x \to x = x$ and by hypothesis $\neg a \to x = x$. Therefore $\neg a \in \{\{\neg a\}^*_L\}^*_R$.
(2) Let $x \in \{a\}^*_R$. Since $\{a\}^*_R \subseteq B(A)$, we can say $\neg x \to x = x$ and $\neg x \to a = a$. Then $a \in \{\neg x : x \in \{a\}^*_R\}^*_L$.
(3) Let $a \in X^*_R \subseteq B(X)$, then we have $a \to x = x$ and $(a \to x) \to a = a$, for all $x \in X$. Hence $x \to a = a$, for all $x \in X$, then $a \in X^*_L$ and $X^* = X^*_R$,
Conversely, let $a \in X^*_R$, then $a \to x = x$, for all $x \in X$. Hence $(a \to x) \to a = x \to a = a$, for all $x \in X$ thus $a \in B(X)$.
(4) By (3) we have $X^*_R \subseteq X^* \subseteq \{0, 1\}$. X^* is proper filter since $\{1\} \neq X$, then $X^* = \{1\}$. Hence $X^* = \{1\}$. □

Proposition 3.13. Let $X \subseteq A$ and $a, b \in A$. Then:
(1) $a/X_R^* = 1/X_R^*$ iff $a \in X_R^*$, hence $a/X_R^* \neq 1/X_R^*$ iff $a \notin X_R^*$,
(2) $a/X_R^* = 0/X_R^*$ iff $\neg a \in X_R^*$,
(3) If X_R^* is proper and $a/X_R^* = 0/X_R^*$, then $a \notin X_R^*$,
(4) $a/X_R^* \leq b/X_R^*$ iff $a \to b \in X_R^*$,
(5) $0 \notin (X, X_R^*)_R^*$ iff $(X, X_R^*)_R^* \neq X_R^*$ be a proper filter of A,
(6) $a/X^* \in X_R^*/X^*$ iff $a \in X_R^*$,
(7) $(X/X^*)_R^* = X_R^*/X^*$.

Proof. We only prove (5), (6) and (7) since the other cases are clear.
(5) Let $0 \notin (X, X_R^*)_R^*$. It is clear that $(X, X_R^*)_R^*$ is a proper filter of A, in contrary, let $(X, X_R^*)_R^* = X_R^*$. Then $(X, X_R^*)_R^*/X_R^*$ is not a proper filter of A, thus there exists $a \in (X, X_R^*)_R^*$ such that $0/X_R^* = a/X_R^*$, by (2) we get that $\neg a \in X_R^*$. Since $X_R^* \subseteq (X, X_R^*)_R^*$, we have $\neg a \in (X, X_R^*)_R^*$. Hence $a * \neg a = 0 \in (X, X_R^*)_R^*$, which is a contradiction.
Conversely, let $(X, X_R^*)_R^* \neq X_R^*$ be proper filter of A. It is clear that $0 \notin (X, X_R^*)_R^*$.
(6) $a/X^* \in X_R^*/X^*$ iff there exists $y \in X_R^*$ such that $a/X^* = y/X^*$ iff there exists $y \in X_R^*$ such that $(a \to y) * (y \to a) \in X^* \subseteq X_R^*$. By Lemma 2.2 $(a \to y) * (y \to a) \leq (y \to a)$. Hence $y \to a \in X_R^*$. Therefore $a \in X_R^*$,
(7) Let $a \in (X/X^*)_R^*$. Then $(a \to x)/X^* = x/X^*$, for all $x/X^* \in X/X^*$, hence $\forall x \in X$, $(a \to x) \to x \in X^* \subseteq X_R^*$, then $((a \to x) \to x) \to x = x$, for all $x \in X$, By Lemma 2.2, $a \to x = x$, for all $x \in X$. Therefore $a \in X_R^*$ and $a/X^* \in X_R^*/X^*$.
Conversely, let $a/X^* \in X_R^*/X^*$. Then $a \in X_R^*$ and by definition $a \to x = x$, for all $x \in X$. Hence $a/X^* \to x/X^* = x/X^*$, for all $x/X^* \in X/X^*$. Therefore $a \in (X/X^*)_R^*$. □

Corollary 3.14. $(\{a^n\}, \{\bigcup y\})_L^* = \{y \in A : a^n \leq y, \exists n \geq 1\}$.

Proof. $y \in (\{a^n\}, \{\bigcup y\})_L^*$ iff $((a^n) \to y) \to y = y$ iff $\exists n \geq 1$ such that $(a^n) \to y = 1$ iff $a^n \leq y$, for some $n \geq 1$. □

Let $1 \neq a \in Reg(A) = \{a \in A : \neg\neg a = a\}$, then we can say $a \notin A^* = \{1\}$.

In the following example we show that $a \notin A^*$ dose not imply $a \in Reg(A)$.

Example 3.15. Consider A in Example 3.5 It is clear that $b \notin A^*$, since $b \notin A_L^*$ but $\neg\neg b = b$ and $b \in Reg(A)$.

Theorem 3.16. Let $X \subseteq A$ and $x \in X_R^*$. Then $\neg\neg x \in X_R^*$.

Proof. Let $X \subseteq A$ and $x \in X_R^*$. By lemma 2.2 $x \leq \neg\neg x$, then for all $a \in A$, $\neg\neg x \to a \leq x \to a = a$ and $a \leq \neg\neg x \to a$. Hence for all $a \in A$, $\neg\neg x \to a = a$. Therefore $\neg\neg x \in X_R^*$.□

In the following example we show that the converse of above theorem is not correct.

Example 3.17. Consider A in Example 3.2, it is clear that $\neg\neg a = 1 \in \{b\}_R^*$, but $a \notin \{b\}_R^* = \{1, b, c\}$.

Corollary 3.18. Let $X \subseteq A$ and $\neg\neg(\neg\neg x \to x) = 1$, for all x in residuated lattice A (Glivenko residuated lattice). Then $X_R^* \subseteq D_s(A)$.

Proof. Let $x \in X_R^*$. By Theorem 3.16, $\neg\neg x \in X_R^*$. Then $(\neg\neg x \to x) = 1$, hence $\neg\neg x = 1$. Therefore $\neg x = 0$ and $x \in D_s(A)$. □

4. Relationship between the stabilizer and filters

Theorem 4.1. Let $X \subseteq A$ and F be a filter of A such that $F \subseteq X_R^*$ and $a/F \in X_R^*/F$. Then $(a \to x) \to x \in F$, for all $x \in X$.

Proof. Let $a/F \in X_R^*/F$. Then there exists $b \in X_R^*$ such that $a/F = b/F$. Hence $(b \to a) * (a \to b) \in F \subseteq X_R^*$. Since $b \to a \in X_R^*$ and $b \in X_R^*$ we have $a \in X_R^*$. Then $a \to x = x$, for all $x \in X$. Therefore $(a \to x) \to x = 1 \in F$. □

Theorem 4.2. Let $X \subseteq A$ and X_R^* be a fantastic filter of A, such that $(A, X_R^*)_R^* \neq \emptyset$. Then $(A, X_R^*)_R^* \subseteq (A, X_R^*)_L^*$.

Proof. At first, we prove if F be a fantastic filter of A and for all $x, y \in A$, $(x \to y) \to y \in F$, we will have $(y \to x) \to x \in F$.

Let $\neg x \in F$ implies $x \in F$ and $(x \to y) \to y \in F$, for all $x, y \in A$. Take $y = 0$, therefore $\neg\neg x = (x \to 0) \to 0 \in F$. Then we can say $x \in F$. By Lemma 2.2, we have $x \leq (y \to x) \to x$. Therefore $(y \to x) \to x \in F$.
By hypothesis we know F is a fantastic filter of A. Then $1 \to (0 \to \neg x) = 1 \in F$ implies $(((\neg x \to 0) \to 0) \to x \in F$, for all $x, y \in A$. Hence $\neg x \to x \in F$. If $\neg x \in F$, then $x \in F$ and therefore $(y \to x) \to x \in F$.
Now, by hypothesis we know $(A, X_R^*)_R^* \neq \emptyset$, then there exists $a \in (A, X_R^*)_R^*$. Hence $(a \to x) \to x \in X_R^*$, for all $x \in X$. Therefore $(x \to a) \to a \in X_R^*$, since X_R^* is a fantastic filter. Then $a \in (A, X_R^*)_L^*$. □

Theorem 4.3. Let F, G be filters of A and F be an obstinate filter of A. Then $(G, F)_R^*$ is an obstinate filter of A.

Proof. We know $(G, F)_R^*$ is a filter of A. If $x, y \notin (G, F)_R^*$ we can say $(x \to a) \to a \notin F$, for all $a \in G$ and $(y \to b) \to b \notin F$, for all $b \in G$. By Lemma 2.2 we have $x \leq (x \to a) \to a$ and $y \leq (y \to b) \to b$, then $x, y \notin F$. F is an obstinate filter thus $x \to y \in F$ and $y \to x \in F$. Then $x \to y = ((x \to y) \to y) \to y \in F$ and $y \to x = ((y \to x) \to x) \to x \in F$, for all $x \in G$. Therefore $(G, F)_R^*$ is an obstinate filter, since $x \to y, y \to x \in (G, F)_R^*$. □

Theorem 4.4. Let F, G be filters of A and F be a Boolean filter. Then $(G, F)_R^*$ is a Boolean filter of A.

Proof. We have to prove $x \vee \neg x \in (G, F)_R^*$, for all $x \in A$. Since F is a Boolean filter we have $x \vee \neg x \in F$, for all $x \in A$. On the other hand we have $x \leq (x \to a) \to a$ and $\neg x \leq (\neg x \to a) \to a$, for all $a \in G$ and we have;

$$\begin{aligned}
x \vee \neg x &\leq (x \to a) \to a \vee (\neg x \to a) \to a, \\
&\leq ((x \to a) \wedge (\neg x \to a)) \to a, \\
&= ((x \vee \neg x) \to a) \to a \in F.
\end{aligned}$$

Therefore $\neg x \vee x \in (G, F)_R^*$, for all $a \in G$. □

Theorem 4.5. Let $X \subseteq A$. Then $Max(A/X^*) = \{N/X^* : N \in Max(A)\}$.

Proof. It is clear that $\mathbf{F}(A/X^*) = \{F/X^* : F \in \mathbf{F}(A)\}$. Let $N \in \mathbf{F}(A)$ such that $X^* \subseteq N$. Then $N \in Max(A)$ iff $N \neq A$ and $N \subseteq F$, ($N = F$, since N is maximal filter of A), for all $F \in \mathbf{F}(A) \backslash \{A\}$ iff $N/X^* = F/X^*$. Therefore $N/X^* \subseteq F/X^*$ (hence $N/X^* = F/X^*$) for all $F/X^* \in \mathbf{F}(A/X^*) \backslash \{A/X^*\}$, iff $N/X^* \in Max(A)$. □

Theorem 4.6. $\{x\}_R^*$ is a prime filter, for all $x \in A$.

Proof. Let $a, b \in A$ such that $a \vee b \in \{x\}_R^*$ and $a, b \notin \{x\}_R^*$. Then $a \to x \neq x$ and $b \to x \neq x$, hence $x < a \to x$ and $x < b \to x$. Since $a \vee b \in \{x\}_R^*$, then we get that $x = (a \vee b) \to x = (b \to x) \wedge (a \to x) < x$, which is a contradiction. □

5. **Conclusion**

In this paper, we introduced the notion of a (right and left) stabilizer and the stabilizer of X with respect to Y in a residuated lattice. Moreover, we presented a characterization and many important properties of the stabilizer and prove X_L^* is filter of A, but X_R^* is not a filter of A. Finally, the relationship between the stabilizer and filters (dense element, $B(X)$) in residuated lattices are studied.

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