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WAVE PROPAGATION THROUGH A NOZZLE WITH ELASTIC WALLS

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Se studiază propagarea micilor perturbații pentru o problemă simplă de curgere-structură. Este considerată curgerea unui fluid izentropic, compresibil, nevâscos printr-o doză cu pereți elastici. În prezența structurii elementului frontieră al fluidului cercetăm influența numărului Mach al mişcării neperturbate asupra vitezei de propagare a undelor.

Study of small perturbations propagation in a simple flow-structure problem shall be made. The flow of a compressible inviscid and isentropic fluid through a nozzle with elastic walls is presented. In presence of a coupling with a structural element bounding the fluid we investigate the influence of Mach number of the unperturbed flow on the speed of propagating waves.

Keywords: Wave propagation, small perturbations, elastic nozzle walls.
Mathematics Subject Classification 2000: 76B15, 74B15, 35Q35.

Introduction

We study the propagation of small perturbations in a nozzle with parallel elastic walls (see [1], [2]). We consider a bi-dimensional inviscid, isentropic and compressible fluid flow through a nozzle with elastic walls. For initial time we suppose that the nozzle has straight walls. The study is divided in two parts. We study in first section the one-dimensional flow and in second the two-dimensional flow-structure interaction.

1. One-dimensional flow-structure problem

Denoting $c(x,t)$, the local speed of sound, $u(x,t)$ the fluid velocity in x direction and $H(x,t)$ the nozzle height, $u_0(x,t)$ shall be the initial fluid velocity.

The lateral section of the nozzle is illustrated in Fig 1.

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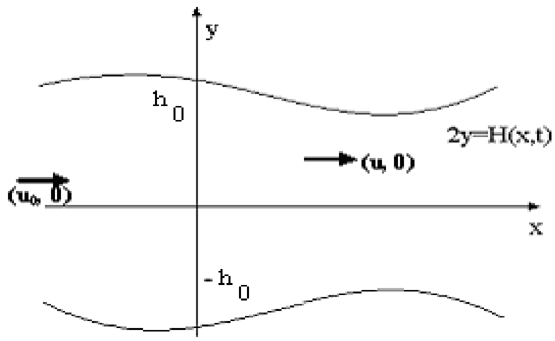


Fig 1. Lateral section of the nozzle.

Under the hypothesis that the walls of the nozzle are so thin that the motion is governed by the linear beam equation (see [1]) the equations governing the flow are:

$$\frac{2\cdot c_t}{\gamma-1}+cu_x+\frac{2\cdot c_x u_0}{\gamma-1}+\frac{c}{H}(H_t+u_0H_x)=0,\ u_t+u_0u_x+\frac{2}{\gamma-1}c_0c_x=0,$$
$$H_{tt}+DH_{xxxx}=m(p_i-p_0), \qquad (1.1)$$

for γ the specific heats ratio, D the bending stiffness, p_i the local pressure of the fluid, p_0 the outside ambient pressure and m the linear mass of the walls that shall be supposed unity.

The evolution of small perturbations for the system (1.1) is expressed by functions c', u', H' for $c = c_0 + c'$, $u = u_0 + u'$, $H = H_0 + H'$. The system for perturbations shall be obtained if we assume that prime quantities are small comparing with those in the unperturbed flow denoted with c_0, u_0, H_0 for $p_0 = 1$, $\rho_0 = 1$. Dropping the prime notation for perturbations and using known relations: $p = p_0(1+(\gamma-1){M_0}^2/2)^{-\gamma/(\gamma-1)}$, $\rho = \rho_0(1+(\gamma-1){M_0}^2/2)^{-1/(\gamma-1)}$, $c_0^2/(\gamma-1) = c^2/(\gamma-1)+u^2/2$ we have for first approximation: $p - p_0 = 2c_0c/(\gamma-1)\cdot \left(c_0^2/\gamma\right)^{1/(\gamma-1)}$ and the system of equations for perturbations is:

$$2c_t/(\gamma-1)+c_0u_x+2c_xu_0/(\gamma-1)+c_0(H_t+u_0H_x)/H_0=0,$$
$$(\gamma-1)(u_t+u_0u_x)+2c_0c_x=0, \qquad (1.2)$$
$$H_{tt}+DH_{xxxx}-2\cdot c_0\left(c_0^2/\gamma\right)^{1/(\gamma-1)}c/(\gamma-1)=0.$$

We search for solutions through simple waves such that for $k,\omega \neq 0$

$$c(x,t)=\varphi(kx-\omega t)\,,\; u(x,t)=\psi(kx-\omega t)\,,\; H(x,t)=h(kx-\omega t)\,, \qquad (1.3)$$

for $\varphi,\psi \in C^{(1)}(\mathbf{R})$, $h \in C^{(4)}(\mathbf{R})$. From (1.2) with $A_0=\left(c_0^2/\gamma\right)^{1/(\gamma-1)}$ we find:

$$2(ku_0-\omega)\varphi'/(\gamma-1)+c_0k\psi'+c_0(ku_0-\omega)h'/H_0=0\,,$$
$$(ku_0-\omega)\psi'+2c_0k\varphi'/(\gamma-1)=0\;. \qquad (1.4)$$

If $ku_0-\omega=0$ then $\varphi=ct.,\psi=ct.,\ h=ct.$ expressing a permanent flow. We shall continue under the case $ku_0-\omega\neq 0$ meaning $\frac{\omega}{k}\neq u_0$ in which phase velocity differs from the flow speed. We can write from (1.4):

$$\psi'=-\frac{c_0k}{(\gamma-1)(ku_0-\omega)}\varphi'\,,\; h'=\frac{(ku_0-\omega)^2}{c_0^2k^2}\varphi'\,,$$
$$Dk^4\frac{(ku_0-\omega)^2}{c_0^2k^2}\varphi^{(4)}-\frac{\omega^2(ku_0-\omega)^2}{c_0^2k^2}\varphi''-\frac{2A_0}{\gamma-1}c_0\varphi=0 \qquad (1.5)$$

Solving differential equation $(1.5)_3$ and denoting $\lambda=\frac{1}{k^2\sqrt{2D}}\sqrt{\omega^2+\frac{\sqrt{\delta}}{|ku_0-\omega|}}$, $\quad \delta=\omega^4+\frac{8Dk^6A_0c_0^3}{\gamma-1}>0$ we shall find the fundamental solutions:

$$\varphi_1=\sin\lambda\xi\,,\ \varphi_2=\cos\lambda\xi\,,\ \varphi_3=\sin\alpha\xi\,,\ \varphi_4=\cos\alpha\xi\,; \qquad (1.6)$$

for $\omega^2>\frac{\sqrt{\delta}}{|ku_0-\omega|},\alpha=\frac{1}{k^2\sqrt{2D}}\sqrt{\omega^2-\frac{\sqrt{\delta}}{|ku_0-\omega|}}$, (1.7)

$$\varphi_1=\sin\lambda\xi\,,\ \varphi_2=\cos\lambda\xi\,,\ \varphi_3=e^{\alpha\xi}\,,\ \varphi_4=e^{-\alpha\xi}\,; \qquad (1.8)$$

for $\omega^2<\frac{\sqrt{\delta}}{|ku_0-\omega|},\alpha=\frac{1}{k^2\sqrt{2D}}\sqrt{\frac{\sqrt{\delta}}{|ku_0-\omega|}-\omega^2}$, (1.9)

or $\varphi_1=\sin\lambda\xi$, $\varphi_2=\cos\lambda\xi$, $\varphi_3=A\xi+B$, for $\omega^2=\frac{\sqrt{\delta}}{|ku_0-\omega|}$. (1.10)

In order to investigate the influence of Mach number of the unperturbed flow on the speed of propagating waves we shall make graphical representations of level curves for the function $f(\omega,k)=\omega^2-\dfrac{\sqrt{\delta}}{|ku_0-\omega|}$. We shall use the following constants: $c_0=\sqrt{1.4}$, $\gamma=1.4025$, D of order 10^{-3}, Mach number $M_0=u_0/c_0\in[0,13]$ and $\omega/k\in[-1,1]$. Representations are made in Figs 2. and 3.

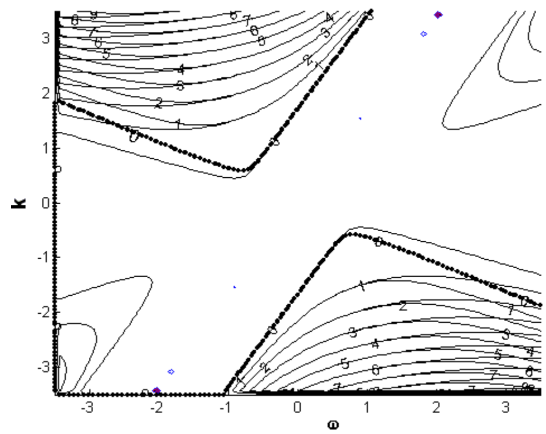
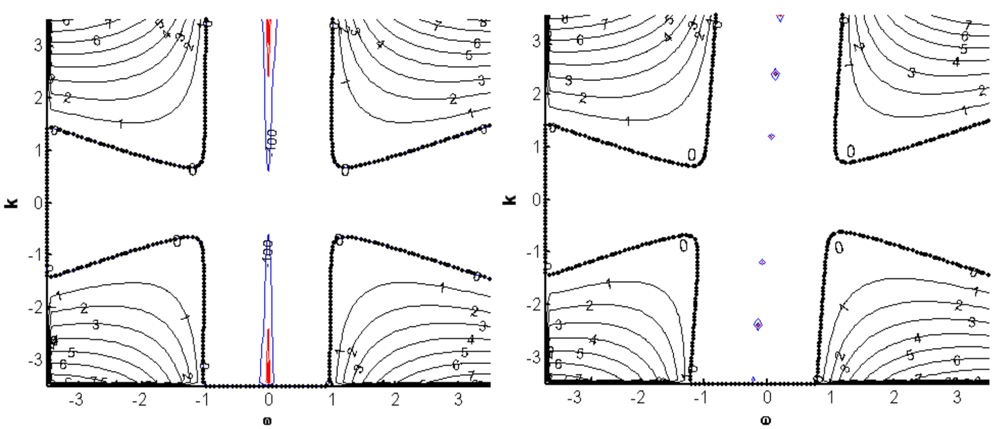
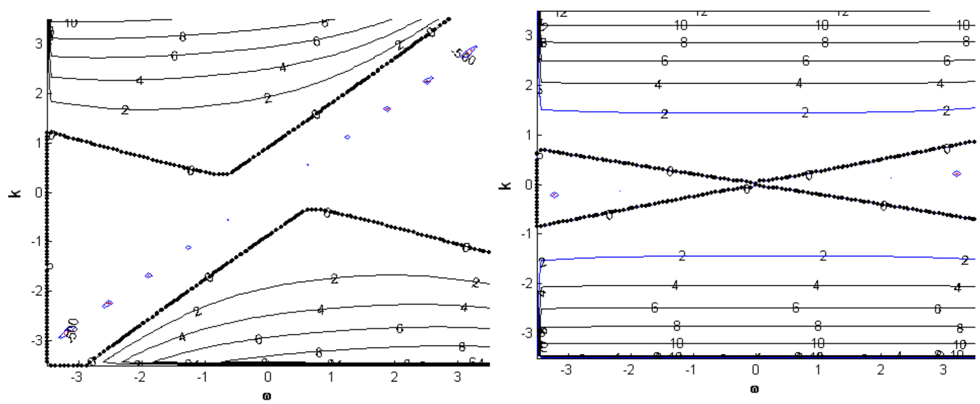


Fig. 2. Variation of D D=0.001, D=0.005 for fixed $M_0=0.5$



a) $M_0=0.005$ b) $M_0=0.05$



c) $M_0 = 0.95$ d) $M_0 = 13$

Fig. 3. Variation of Mach number for fixed D=0.001.

From Figs 2. and 3. we conclude that the domain $\omega / k < 0$ reduces once Mach number is increasing.

For $\xi = kx - \omega t$ we found bounded general solution for $(1.3)_1$:

$$c(x,t) = F_1 \sin \lambda (kx - \omega t) + F_2 \cos \lambda (kx - \omega t) + F_3 \sin \alpha (kx - \omega t) + \\ + F_4 \cos \alpha (kx - \omega t), \tag{1.11}$$

for $\omega^2 > \dfrac{\sqrt{\delta}}{|ku_0 - \omega|}$, and in case $\omega^2 < \dfrac{\sqrt{\delta}}{|ku_0 - \omega|}$ the local speed of sound becomes:

$$c(x,t) = F_1 \sin \lambda (kx - \omega t) + F_2 \cos \lambda (kx - \omega t), \ \frac{\omega}{k} > 0,$$

$$c(x,t) = F_1 \sin \lambda (kx - \omega t) + F_2 \cos \lambda (kx - \omega t) + F_3 e^{\alpha(-kx + \omega t)}, k > 0 > \omega \tag{1.12}$$

$$c(x,t) = F_1 \sin \lambda (kx - \omega t) + F_2 \cos \lambda (kx - \omega t) + F_3 e^{\alpha(kx - \omega t)}, \ k < 0 < \omega.$$

Then from (1.5) after integration we can write:

$$H(x,t) = \frac{(ku_0 - \omega)^2}{c_0^2 k^2} c(x,t) + C_1; \ u(x,t) = -\frac{c_0 k}{(\gamma - 1)(ku_0 - \omega)} c(x,t) + C_2. \tag{1.13}$$

Using initial conditions: $c(x,t) = c_0 C(x)$, $u(x,t) = u_0 U(x)$, $H(x,t) = H_0 f(x)$, and boundary conditions we can express the constants of integration. Also from these conditions one obtains a *compatibility relation between initial conditions* for existence of motion through simple waves: $\dfrac{(ku_0 - \omega)^2}{c_0^2 k^2} c_0 C(x) + C_1 = H_0 f(x)$,

$-\dfrac{c_0 k}{(\gamma - 1)(ku_0 - \omega)} c_0 C(x) + C_2 = u_0 U(x)$, from where:

$$u_0 U(x) - C_2 = c_0{}^3 k^3 /(\gamma - 1)(ku_0 - \omega)^3 (C_1 - H_0 f(x)). \quad (1.14)$$

Remarks:
1. If $U(x) = ct.$ or $f(x) = ct.$ then $f(x) = ct., C(x) = ct.$ or $U(x) = ct., C(x) = ct.$ Looking for solution which has: $c(x,t) = c_o\varepsilon$, $u(x,t) = 0$, $H(x,t) = H_0\varepsilon$ or $c(x,t) = c_o\varepsilon$, $u(x,t) = u_0\varepsilon$, $H(x,t) = 0$, $\varepsilon = O(10^{-2})$ we find

$$F_1 \sin \lambda kx + F_2 \cos \lambda kx + F_3 \sin \alpha kx + F_4 \cos \alpha kx = c_0 C,$$

$$\frac{(ku_0 - \omega)^2}{c_0^2 k^2} c_0 C + C_1 = H_0 \varepsilon, \frac{c_0 k}{(\gamma - 1)(ku_0 - \omega)} c_0 C + C_2 = u_0 \varepsilon.$$

2. *Boundary conditions* could be imposed if we consider the domain $x \in [0, L]$, $y \in [0, H(x)]$ at each t: $H(0,t) = H(L,t) = 0$, $H_{xx}(0,t) = H_{xx}(L,t) = 0$.

Considering also the case $f(x) \neq ct., U(x) \neq ct.$ we look for the constants F_i, i=1,2,3,4 in order to obtain the general solution.

2. Bi-dimensional flow-structure problem

The fluid velocity has now two components, on x and y direction. The fluid flow is sketched in Fig. 4.

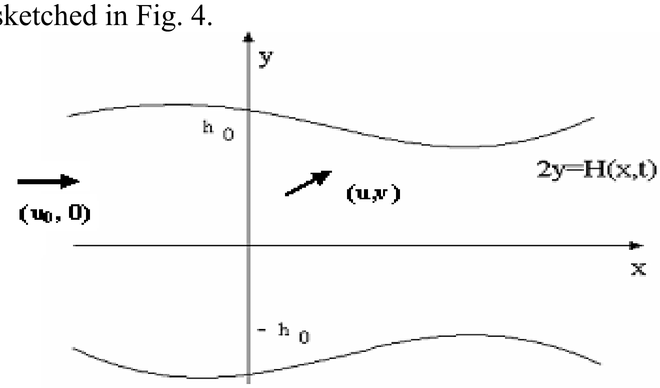


Fig. 4. Lateral section of the nozzle.

We consider a potential flow, with the potential $\phi(x, y, t)$ and flow velocity vector $\vec{V} = u_0 \vec{i} + \vec{v}, \vec{v} = (u, v) = (\phi_x, \phi_y)$. The equations governing the flow are:

$$(1 - M_0^2)\phi_{xx} + \phi_{yy} - (2u_0\phi_{xt} + \phi_{tt}) / c_0{}^2 = 0,$$
$$H_{tt} + DH_{xxxx} = p_i - p_0 = 2\rho_0(\phi_t + u_0\phi_x) \text{ on } 2y = H(x,t), \quad (2.1)$$

where $(u_0,0)$ is the velocity vector in the unperturbed flow and ϕ, H bounded functions for $y \to \pm\infty$ (see [3], [4], [5]).

Remarks:
1. For bi-dimensional inviscid, isentropic and compressible fluid the pressure and mass density could be written as: $p = p_0(1+(\gamma-1)(\phi_t + V^2/2)/c_0^2)^{\gamma/(\gamma-1)}$, $\rho = \rho_0(1+(\gamma-1)(\phi_t + V^2/2)/c_0^2)^{1/(\gamma-1)}$ with the module of velocity: $V^2 = (u_0+u)^2 + v^2$. Then for small perturbations $V^2 = u_0{}^2 + 2u_0 u$ and pressure on $2y = H(x,t)$ becomes: $p = p_0(1-\gamma(\phi_t + \frac{1}{2}2u_0 u)/c_0^2 + \ldots)$ from where $p_0 - p = \rho_0(\phi_t + u_0\phi_x)$.
2. Boundary condition on the surface $2y = H(x,t)$ must be imposed for potential obtained from velocity in x direction: $\phi_y = H_t + u_0 H_x$.

For function H we shall consider initial and boundary conditions:
$H(x,0) = H_0 f(x)$, and $H(0,t) = H(L,t) = 0$, $H_{xx}(0,t) = H_{xx}(L,t) = 0$.

Looking for motion ***through simple waves*** we consider:
$\phi(x,y,t) = F(k_1 x + k_2 y - \omega t)$, $H(x,t) = h(k_1 x - \omega t)$, $F \in C^{(2)}(\mathbf{R})$, $h \in C^{(4)}(\mathbf{R})$.

From (2.1) we find:

$$(k_1^2 + k_2{}^2 - (\omega/c_0 - M_0 k_1)^2)F'' = 0,$$

$$\omega^2 h'' + Dk_1^4 h^{(4)} = 2\rho_0(k_1 u_0 - \omega)F'. \tag{2.2}$$

For $F'' \neq 0$ one find the ***dispersion equation***:

$$k_2 = \pm\sqrt{(\frac{\omega}{c_0} - M_0 k_1)^2 - k_1^2} == \pm\sqrt{\frac{(\omega - k_1 u_0)^2}{c_0^2} - k_1^2}.$$

In order to solve equation $(2.1)_1$ we change the variables (x,t) through (ξ,η) with: $\xi = x - (u_0 + c_0)t$, $\eta = x - (u_0 - c_0)t$ obtaining a new equation:

$$4\phi_{\xi\eta} + \phi_{yy} = 0, \quad \phi = \phi(\xi,\eta,y). \tag{2.3}$$

that will be solved considering a separation of variables (ξ,η) from y variables: $\phi=\varphi(\xi,\eta)F(y)$. We find $\frac{\varphi_{\xi\eta}}{\varphi}=-\frac{1}{4}\frac{F''(y)}{F(y)}=K$, $K>0$ for bounded solutions on y.

Then $F(y)=A\sin 2\lambda y+B\cos 2\lambda y$ and $\varphi_{\xi\eta}-\lambda^2\varphi=0$. Solution obtained for $\lambda=0$ is $\varphi(\xi,\eta)=C\xi+D\eta$ and for $\lambda\neq 0$ is $\varphi(\xi,\eta)=$ $=Ce^{\lambda(\xi+\eta)}+De^{-\lambda(\xi+\eta)}$. We can write the solution of $(2.1)_1$:

$$\phi(x,y,t)=[Ce^{2\lambda(x-u_0t)}+De^{-2\lambda(x-u_0t)}][A\sin 2\lambda y+B\cos 2\lambda y]. \quad (2.4)$$

We remark that for solution (2.4) of $(2.1)_1$ we have $\phi_t+u_0\phi_x=0$.

Solving equation $(2.1)_2$ we obtain for function $H(x,t)$ the problem:

$$\begin{aligned}&H_{tt}+DH_{xxxx}=0\,,\ D=[0,L]\times(0,\infty)\,,\\&H(x,0)=H_0f(x),\\&H(0,t)=H(L,t)=0,\ H_{xx}(0,t)=H_{xx}(L,t)=0.\end{aligned} \quad (2.5)$$

Considering a separation of variables $H(x,t)=X(x)T(t)\Rightarrow\frac{T''}{T}=-D\frac{X^{(4)}}{X}=\alpha$ and for $\alpha>0$ we can write $T''-\alpha T=0,\ X^{(4)}+(\alpha/D)X=0$.

With $\beta=\sqrt[4]{\alpha/D}/\sqrt{2}$, general solution for (2.5) is:

$X(x)=\cos\beta x\cdot(C_1e^{\beta x}+C_3e^{-\beta x})+\sin\beta x\cdot(C_2e^{\beta x}+C_4e^{-\beta x})$,

$T(t)=Pe^{\sqrt{\alpha}t}+Qe^{-\sqrt{\alpha}t}$, $P=0$, for a bounded solution on t. (2.6)

Solution for equation $(2.5)_1$ is:

$$H(x,t)=\{\cos\beta x[C_1e^{\beta x}+C_3e^{-\beta x}]+\sin\beta x[C_2e^{\beta x}-C_4e^{-\beta x}]\}Qe^{-\sqrt{\alpha}t} \quad (2.7)$$

From conditions $(2.5)_2$ we find:

$$\begin{aligned}&H(0,t)=0\Rightarrow C_3=-C_1, H_{xx}(0,t)=0\Rightarrow C_2=C_4,\\&H(L,t)=0\Rightarrow C_1\cos\beta L(e^{\beta L}-e^{-\beta L})+\sin\beta L(C_2e^{\beta L}+C_4e^{-\beta L})=0,\\&H_{xx}(L,t)=0\Rightarrow 2\beta L=-\frac{\pi}{2}+2k\pi, k\in N\Rightarrow\sqrt{\alpha}=\sqrt{2D}(-\pi/4+k\pi)^2/L^2,\end{aligned} \quad (2.8)$$

and (2.7) $\Rightarrow C_2 = C_1 th(-\pi/4 + k\pi)$ for each k.

The general solution for (2.5) is:

$$H(x,t) = \sum_{k=1}^{\infty} M_k \{\cos((-\pi/4 + k\pi)x/L) sh((-\pi/4 + k\pi)x/L) + \tag{2.9}$$

$$+ th(-\pi/4 + k\pi)\sin((-\pi/4 + k\pi)x/L) ch((-\pi/4 + k\pi)x/L)\} e^{-\frac{\sqrt{2D}}{L^2}(-\frac{\pi}{4} + k\pi)^2 t} .$$

with M_k determined from $H(x,0) = H_0 f(x)$ for which:

$$H_0 f(x) = \sum_{k=1}^{\infty} M_k \{\cos((-\pi/4 + k\pi)x/L) sh((-\pi/4 + k\pi)x/L) + \tag{2.10}$$

$$+ th(-\pi/4 + k\pi)\sin((-\pi/4 + k\pi)x/L) ch((-\pi/4 + k\pi)x/L)\} .$$

Conclusions

For one-dimensional case was investigated the influence of Mach number of the unperturbed flow upon the speed of propagating waves (presented in Figs 2 and 3).

For the bi-dimensional case instead of a discussion of the dispersion equation we have studied the solution for potential using condition on the boundary: $\phi_y = H_t + u_0 H_x$ on $2y = H(x,t)$, and initial conditions on velocity. The velocity field is expressed by:

$$u = \phi_x = 2\lambda[Ce^{2\lambda(x-u_0 t)} - De^{-2\lambda(x-u_0 t)}][A\sin 2\lambda y + B\cos 2\lambda y],$$
$$v = \phi_y = 2\lambda[Ce^{2\lambda(x-u_0 t)} + De^{-2\lambda(x-u_0 t)}][A\cos 2\lambda y - B\sin 2\lambda y].$$

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