SPECTRAL EQUIVALENCE OF S-SPECTRAL OPERATORS

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În această lucrare studiem proprietatea de spectral echivalenţă a unei clase speciale de operatori, operatorii $S-$spectrali (definiţi şi introduşi de I. Bacalu în [5], [6], [7]), *punând în evidenţă legătura dintre acest tip de operatori şi această proprietate. Una dintre trăsăturile importante ale familiei operatorilor decompozabili (spectrali, scalari, spectrali (scalari) generalizaţi, A-scalari) este transportul proprietăţilor spectrale de la un operator la altul prin intermediul echivalenţei spectrale* ([9], [10]). *În cazul familiei operatorilor $S-$spectrali, majoritatea proprietăţilor spectrale se păstrează. Precum se ştie, pentru operatorii spectrali T_1, T_2 spectral echivalenţi, proprietăţile spectrale ale lui T_1 se transferă la T_2. Pentru operatorii $S-$spectrali, acest rezultat rămâne parţial adevărat, dar spectral echivalenţa nu este "echivalentă" cu egalitatea $S-$măsurilor spectrale; această egalitate implică o proprietate mai slabă numită echivalenţă $S-$spectrală, care de fapt este naturală în acest caz.*

In this paper, we study the spectral equivalence property of a special class of operators, namely $S-$spectral operators (defined and introduced by I. Bacalu in [5], [6], [7]), *highlighting the link between $S-$spectral operators and spectral equivalence. One of the essential characteristics of the class of decomposable (spectral, scalar, generalized spectral (scalar), $A-$scalar) operators is the transfer of the spectral proprieties from one operator to another using spectral equivalence* ([9], [10]). *The family of $S-$spectral operators preserves the most interesting properties of spectral operators. As is known, for the spectral operators T_1, T_2 which are spectral equivalent, the spectral properties of T_1 transfer to T_2. For $S-$spectral operators, this fact remains partially true, but the spectral equivalence is not "equivalent" to equality of $S-$spectral measures; this equality involves only a weaker property called $S-$spectral equivalence, which is natural in this case.*

Keywords: spectral space; spectral ($S-$spectral) measure; spectral ($S-$spectral) operator; spectral ($S-$spectral) equivalence.

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1. Introduction

Let X be a Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X, let $\mathcal{P}_X$ be the set of all projectors on X and let $\mathfrak{B}_S$ be the family of all Borelian sets B of the complex plane $\mathbb{C}$ which have the property that $B \cap S = \varnothing$ or

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$B \supset S$, where S is a compact set of $\mathbb{C}$. If $T \in \mathbf{B}(X)$ and Y is a linear (closed) subspace of X invariant to T, let us denote by $T \mid Y$ the restriction of T to Y and by $\dot{T}$ the operator induced by T in the quotient space $\dot{X} = X / Y$. We also denote by $\sigma(T)$ ($\rho(T) = \mathbb{C} \setminus \sigma(T)$) the spectrum (the resolvent set) of T and by $R(\lambda, T) = (\lambda I - T)^{-1}$, with $\lambda \in \rho(T)$, the resolvent of T.

An operator $T \in \mathbf{B}(X)$ is said to have the *single-valued extension property* (SVEP) if for any analytic function $f : D_f \to X$ (where $D_f \subset \mathbb{C}$ is an open set), with $(\lambda I - T) f(\lambda) \equiv 0$ it results that $f(\lambda) \equiv 0$ ([9], [10], [12]).

For an operator $T \in \mathbf{B}(X)$ that has SVEP and for $x \in X$, we consider the set $\rho_T(x)$ of all elements $\lambda_0 \in \mathbb{C}$ such that there is a X-valued analytic function $\lambda \to x(\lambda)$ defined in a neighborhood of λ_0 which verifies the relation $(\lambda I - T) x(\lambda) \equiv x$; $x(\lambda)$ is unique, $\rho_T(x)$ is open and $\rho(T) \subset \rho_T(x)$. We take $\sigma_T(x) = \complement \rho_T(x) = \mathbb{C} \setminus \rho_T(x)$ and $X_T(F) = \{x \in X; \sigma_T(x) \subset F\}$, where $F \subset \mathbb{C}$ is closed. $\rho_T(x)$ is called the *local resolvent set of* x *with respect to* T and $\sigma_T(x)$ is the *local spectrum of* x *with respect to* T ([3], [9], [17]).

Let $T \in \mathbf{B}(X)$ and let Y be a closed subspace of X. We recall that Y is a *spectral maximal space* of T if it is an invariant subspace to T such that for any other subspace Z of X, invariant to T, the inclusion $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ implies the inclusion $Z \subset Y$ ([10], [13]).

An operator $T \in \mathbf{B}(X)$ is *decomposable* if for any finite open covering $\{G_i\}_{i=1}^{n}$ of $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^{n}$ of spectral maximal spaces of T such that $\sigma(T \mid Y_i) \subset G_i$ $(i = 1, 2, ..., n)$ and $X = Y_1 + Y_2 + ... + Y_n$ ([10], [13]). An operator $T \in \mathbf{B}(X)$ is *strongly decomposable* if $T \mid Y$ is decomposable for any spectral maximal space Y of X ([3], [4]).

In order to study the link between $S-$spectral operator and spectral equivalence, we need several notions from the theory of residually spectral decompositions brought up by F.H. Vasilescu in [17], [18], [19].

An open set $\Omega \subset \mathbb{C}$ is said to be a *set of analytic uniqueness* for $T \in \mathbf{B}(X)$ if for any open set $\omega \subset \Omega$ and any analytic function $f_0 : \omega \to X$ satisfying the equation $(\lambda I - T) f_0(\lambda) \equiv 0$, it follows that $f_0(\lambda) \equiv 0$ in ω. For $T \in \mathbf{B}(X)$ there is a unique *maximal open set* Ω_T *of analytic uniqueness* ([17], 2.1.). We shall denote by $S_T = \complement\Omega_T = \mathbb{C}\setminus\Omega_T$ and call it the *analytic spectral residuum of* T ([17], [19]).

For $x\in X$, a point λ is in $\delta_T(x)$ if in a neighborhood V_λ of λ there is at least an analytic function f_x (called T-*associated to* x) such that $(\mu I-T)f_x(\mu)\equiv x$, for all $\mu\in V_\lambda$. We shall put

$$\gamma_T(x)=\complement\delta_T(x),\ \rho_T(x)=\delta_T(x)\cap\Omega_T,\ \sigma_T(x)=\complement\rho_T(x)=\gamma_T(x)\cup S_T \text{ and}$$
$$X_T(F)=\{x\in X;\ \sigma_T(x)\subset F\}$$

where $S_T\subset F\subset\mathbb{C}$ ([17]).

An operator $T\in\mathbf{B}(X)$ has SVEP if and only if $S_T=\varnothing$; then we have $\sigma_T(x)=\gamma_T(x)$ and there is in $\rho_T(x)=\delta_T(x)$ a unique analytic function $x(\lambda)$, $T-$associated to x, for any $x\in X$. We shall recall that if $T\in\mathbf{B}(X)$, $S_T\neq\varnothing$ and $X_T(F)$ is closed, for $F\subset\mathbb{C}$ closed, $S_T\subset F$, then $X_T(F)$ is a spectral maximal space of T and $\sigma(T\mid X_T(F))\subset F$ ([17], Propositions 2.4. and 3.4.).

2. Preliminaries

Definition 2.1. A family of open sets $G_S\cup\{G_i\}_{i=1}^n$ is said to be an $S-$*covering* of the closed set $\sigma\subset\mathbb{C}$ if $G_S\cup\left(\bigcup_{i=1}^n G_i\right)\supset\sigma\cup S$ and $\overline{G}_i\cap S=\varnothing$ $(i=1,2,...,n)$ (where $S\subset\mathbb{C}$ is also closed) ([17]).

Definition 2.2. Let $T\in\mathbf{B}(X)$ and let $S\subset\sigma(T)$ be a compact set. T is called $S-$*decomposable* (respectively, *strongly* $S-$*decomposable*) if for any finite open $S-$covering $G_S\cup\{G_i\}_{i=1}^n$ of $\sigma(T)$, there is a system $Y_S\cup\{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that

i) $\sigma(T\mid Y_S)\subset G_S$, $\sigma(T\mid Y_i)\subset G_i$ $(i=1,2,...,n)$,

ii) $X=Y_S+\sum_{i=1}^n Y_i$ (respectively, $Z=(Z\cap Y_S)+\sum_{i=1}^n(Z\cap Y_i)$, where Z is any spectral maximal space of T) ([5], [6], [8]).

Obviously, for $S=\varnothing$ we obtain a decomposable operator ([13]), respectively a strongly decomposable operator ([4]).

Definition 2.3. For $T_1, T_2 \in \mathbf{B}(X)$, not necessarily permutable, we use the notation $(T_1 - T_2)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_1^k T_2^{n-k}$.

We say that $T_1, T_2 \in \mathbf{B}(X)$ are *spectral equivalent* (or *quasi-nilpotent equivalen*t) ([9], [10]) and write $T_1 \sim T_2$ if

$$\lim_{n\to\infty} \left\| (T_1 - T_2)^{[n]} \right\|^{\frac{1}{n}} = 0 \text{ and } \lim_{n\to\infty} \left\| (T_2 - T_1)^{[n]} \right\|^{\frac{1}{n}} = 0 .$$

This relation is reflexive, symmetric and transitive.

To show the relevance and the necessity of studying the spectral equivalence property for the family of $S-$decomposable operators (in particular, $S-$spectral operators), we emphasize the consistency of this class, in the sense of how many and varied are the subfamilies that compose it: the restrictions and the quotients (with respect to an invariant subspace or a spectral maximal space) of decomposable (unitary, self-adjoint, normal, spectral (scalar), generalized spectral (scalar), A-scalar, A-unitary) operators are $S-$decomposable; the perturbations and the direct sums composed by one decomposable operator and another operator are $S-$decomposable; the subdecomposable (subnormal, subscalar) operators are $S-$decomposable (practically, S-normal, S-scalar), as restrictions of decomposable (normal, scalar) operators; the quasinormal operators (i.e. T commutes with T^*T), being subnormal, are $S-$decomposable; for cosubnormal operators (i.e. T^* is subnormal), the adjointable operators T^* are S-decomposable; Cesaro operators are subscalar, hence $S-$scalar; the operators which admit scalar dilatations or $A-$scalar dilatations are $S-$decomposable. M. Putinar showed that the hiponormal operators are subscalar, hence S-decomposable ([16]). In fact, E.J. Albrecht and J. Eschmeier showed that any operator is the quotient of a restriction or the restriction of a quotient of decomposable operators ([1]), thus any operator is $S-$decomposable or similar to an $S-$decomposable operator.

In what follows, we mention briefly the important results for decomposable and spectral operators, respectively for $S-$decomposable operators $T_1, T_2 \in \mathbf{B}(X)$: if $T_1 \sim T_2$, then $\sigma(T_1) = \sigma(T_2)$ and $S_{T_1} = S_{T_2}$ ([9], [10], [7]); if $T_1 \sim T_2$ and $S_{T_1} \neq \emptyset, S_{T_2} \neq \emptyset$, then $\gamma_{T_1}(x) = \gamma_{T_2}(x)$, for $x \in X$ ([7]); if $T_1 \sim T_2$ and T_1 has SVEP, then T_2 has SVEP and $\sigma_{T_1}(x) = \sigma_{T_2}(x)$, for $x \in X$ ([9], [10]); if $T_1 \sim T_2$ and T_1 is decomposable (spectral), then T_2 is decomposable (spectral) and $X_{T_1}(F) = X_{T_2}(F)$, for $F \subset \mathbb{C}$ closed ([9], [10]); if $T_1 \sim T_2$ and T_1 is $S-$decomposable (strongly $S-$decomposable), then T_2 is S-decomposable (respectively, strongly $S-$decomposable) and $X_{T_1}(F) = \,= X_{T_2}(F)$, for $F \subset \mathbb{C}$, $F \supset S$ closed ([7]); if T_1, T_2 are decomposable (respectively, spectral with E_1, E_2 their spectral measures), then T_1 and T_2 are spectral equivalent if and only if $X_{T_1}(F) = X_{T_2}(F)$, for $F \subset \mathbb{C}$ closed (respectively, $E_1 = E_2$) ([9], [10]).

3. Spectral equivalence of the $S-$spectral operators

Definition 3.1. An application $E_S : \mathfrak{B}_S \to \mathcal{P}_X$ is said to be an $S-$*spectral measure* if

1) $E_S(\mathbb{C}) = I,\ E_S(\varnothing) = 0$
2) $E_S(B_1 \cap B_2) = E_S(B_1)E_S(B_2),\ B_1, B_2 \in \mathfrak{B}_S$
3) $E_S\left(\bigcup_{m=1}^{\infty} B_m\right)x = \sum_{m=1}^{\infty} E_S(B_m)x,\ B_m \in \mathfrak{B}_S, B_p \cap B_m = \varnothing,\ m \neq p$
4) $\sup_{B \in \mathfrak{B}_S} \left\| E_S(B) \right\| < \infty$.

An operator $T \in \mathbf{B}(X)$ will be said to be $S-$*spectral* if there is E_S an $S-$spectral measure such that the following conditions are verified:

5) $TE_S(B) = E_S(B)T,\ B \in \mathfrak{B}_S$
6) $\sigma(T \mid E_S(B)X) \subset \overline{B},\ B \in \mathfrak{B}_S$.

For $S = \varnothing$, we obtain a spectral measure and a spectral operator ([12]).

Remark 3.1. An operator $T \in \mathbf{B}(X)$ is $S-$spectral if and only if it is a direct sum $T = T_1 \oplus T_2$, where T_1 is spectral and $\sigma(T_2) \subset S$.

Proof. Indeed, if T is $S-$spectral, then one can easily verify that the map $E : \mathfrak{B} \to \mathcal{P}_X$ (where $\mathfrak{B} = \mathfrak{B}_\varnothing$) defined by $E(B) = E_S(B \cap \complement S),\ B \in \mathfrak{B}$ is a spectral measure for $T_1 = T \mid E_S(\complement S)X$, hence $T = T_1 \oplus T_2$, where $T_2 = \,= T \mid E_S(S)X$ and $\sigma(T_2) \subset S$.

Conversely, if $T_1 \in \mathbf{B}(X_1)$ is spectral and $T_2 \in \mathbf{B}(X_2)$ is not spectral, with $\sigma(T_1) \not\subset \sigma(T_2)$, by putting $S = \sigma(T_2)$, $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$, it follows that the map $E_S : \mathfrak{B}_S \to \mathcal{P}_X$ defined by the equalities $E_S(B) = \,= E(B) \oplus 0$, if $B \cap S = \varnothing,\ B \in \mathfrak{B}_S$ and $E_S(B) = E(B) \oplus I_2$, if $B \supset S,\ B \in \mathfrak{B}_S$, is an $S-$spectral measure of T (where E is the spectral measure of T_1 and I_2 is the identity operator in X_2).

Lemma 3.1. *Let* $T \in \mathbf{B}(X)$, *let* Y *be an invariant subspace to* T *and let* $\dot{T}$ *be the operator induced by* T *in the quotient space* $\dot{X} = X / Y$. *If* T *and* $\dot{T}$ *have SVEP, then* $X_T\left(\sigma(T \mid Y) \setminus \sigma\left(\dot{T}\right)\right) \subset Y$.

Proof. If $x \in X_T\left(\sigma(T \mid Y) \setminus \sigma\left(\dot{T}\right)\right)$, we have $\sigma_T(x) \subset \sigma(T \mid Y) \setminus \sigma\left(\dot{T}\right)$. Because the equalities $S_T = \varnothing$ and $S_{\dot{T}} = \varnothing$ imply the relations $\gamma_T(x) = \sigma_T(x)$, $\gamma_{\dot{T}}(\dot{x}) = \sigma_{\dot{T}}(\dot{x})$ and $\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x)$ ([7], Proposition 2.1.), then it follows that $\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x) \cap \sigma\left(\dot{T}\right) \subset \left(\sigma(T \mid Y) \setminus \sigma\left(\dot{T}\right)\right) \cap \sigma\left(\dot{T}\right) = \varnothing$, hence $\dot{x} = \dot{0}$, whence $x \in Y$.

In the following propositions, many examples of $S-$spectral operators can be obtained in general conditions, namely the restrictions and the quotients of spectral operators with respect to an invariant subspace.

Proposition 3.1. *Let* $T \in \mathbf{B}(X)$ *be a spectral* (*scalar*) *operator having the spectral measure* E, *let* Y *be a closed linear subspace invariant to* T, *let* $\dot{T}$ *be the operator induced by* T *in* $\dot{X} = X / Y$ *and let* $\varphi : X \to \dot{X}$ *be the canonical application. Then* $\dot{T} = \dot{T}_1 \oplus \dot{T}_2$, *where* $\dot{T}_1 = \dot{T} \mid \varphi\left(E(\sigma')X\right)$ *is spectral* (*scalar*) *operator,* $\dot{T}_2 = \dot{T} \mid \varphi\left(E(\sigma)X\right)$, $\sigma = \sigma(T \mid Y)$, $\sigma' = \sigma\left(\dot{T}\right) \setminus \sigma(T \mid Y)$ *and* $\sigma\left(\dot{T}_2\right) \subset S = \sigma(T \mid Y) \cap \sigma\left(\dot{T}\right)$.

Proof. The operator $T \mid E(\sigma')X$ is spectral (scalar) ([12], III, XV, 16) and since $Y \subset E(\sigma)X = X_T(\sigma)$, we have $Y \cap E(\sigma')X = \{0\}$. But $E(\sigma')X + Y = E(\sigma')X \oplus Y$ (because $E(\sigma')X + Y$ is closed; see [7], Lemma 1.1.13.), hence $\varphi\left(E(\sigma')X\right)$ can be identified with $E(\sigma')X$ and $\dot{T}_1$ with $T \mid E(\sigma')X$ ([7], Remark 1.1.15.), therefore $\dot{T}_1$ is spectral (scalar). It is easily to verify that $\varphi\left(X_T(\sigma)\right) = \dot{X}_{\dot{T}}(\sigma) = \dot{X}_{\dot{T}}(S)$ is a spectral maximal space of $\dot{T}$ ([7], Theorem 1.1.19. and Corollary 1.1.20.), consequently $\sigma\left(\dot{T}_2\right) = \sigma\left(\dot{T} \mid \varphi\left(X_T(\sigma)\right)\right) \subset S$.

Proposition 3.2. *Let* $T \in \mathbf{B}(X)$ *be a spectral* (*scalar*) *operator with its spectral measure* E, *let* Y *be an invariant subspace to* T, *let* $\dot{T}$ *be the operator induced by* T *in* $\dot{X} = X / Y$, *with* $X_T(\sigma) \subset Y$ (*where* $\sigma = \sigma(T \mid Y) \setminus \sigma\left(\dot{T}\right)$). *Let also*

$S=\sigma(T\,|\,Y)\bigcap\sigma(\dot{T})$ *and let* $T_Y=T\,|\,Y$. *Then* $T_Y\,|\,E(\sigma)Y$ *and* $T_Y\,|\,\overline{X_T(\sigma)}$ *are spectral* (*scalar*) *operators and* $T_Y=(T_Y\,|\,E(\sigma)Y)\oplus(T_Y\,|\,E(S)Y)$, *where* $\sigma(T_Y\,|\,E(S)Y)\subset\widetilde{S}\bigcap\sigma(T_Y)$.

Proof. The set σ being open in $\sigma(T)$, there is a growing sequence of open sets $(\sigma_n)_{n\in\mathbb{N}}$ with $\sigma=\bigcup_{n\in\mathbb{N}}\sigma_n$; from the continuity of the measures $E(\cdot)x$ it results that $E(\sigma)=\lim_{n\to\infty}E(\sigma_n)$, therefore $E(\sigma_n)X=X_T(\sigma_n)\subset X_T(\sigma)$ implies $E(\sigma)X\subset\overline{X_T(\sigma)}\subset Y$. The closed subspaces $E(\sigma)X$ and $\overline{X_T(\sigma)}$ are invariant to T and to spectral measure E, hence $T_Y\,|\,E(\sigma)Y$ and $T_Y\,|\,\overline{X_T(\sigma)}$ are spectral (scalar) operators ([11]). From $Y\subset X_T(\sigma(T|Y))=E(\sigma(T|Y))X$ it follows that $Y=E(\sigma(T|Y))Y=E(\sigma)Y+E(S)Y$, hence Y is invariant to $E(\sigma)$ and $E(S)$; consequently $E(\sigma)|Y$ and $E(S)|Y$ are projectors in Y, $E(\sigma)Y$ and $E(S)Y$ are closed subspaces and $Y=E(\sigma)Y\oplus E(S)Y$. We also obtain that

$$T_Y=(T_Y\,|\,E(\sigma)Y)\oplus(T_Y\,|\,E(S)Y) \text{ and}$$

$$\sigma(T_Y\,|\,E(S)Y)\subset\widetilde{\sigma(T\,|\,E(S)X)}\bigcap\sigma(T\,|\,Y)\subset\widetilde{S}\bigcap\sigma(T\,|\,Y),$$

where $\widetilde{S}=\mathbb{C}\setminus D^{\infty}$, D^{∞} being the unbounded component of $\mathbb{C}\setminus S$, equivalent to the condition as $\complement S=\mathbb{C}\setminus S$ to be connected.

Theorem 3.1. *Let* $T\in\mathbf{B}(X)$ *be a spectral operator and let* Y *be a closed subspace invariant to* T *such that* $X_T(\sigma)\subset Y$, *where* $\sigma=\sigma(T\,|\,Y)\setminus\sigma(\dot{T})$ *and* $S=\widetilde{S}$, *where* $S=\sigma(T\,|\,Y)\bigcap\sigma(\dot{T})$. *Then* $T\,|\,Y$ *and* $\dot{T}$ *are* $S-$*spectral operators.*

Proof. These assertions follow by Propositions 3.1., 3.2. and Remark 3.1.

Theorem 3.2. *Let* $T\in\mathbf{B}(X)$ *be an* $S-$*spectral operator and let* E_S *be its* $S-$*spectral measure. Then for any* $F\subset\mathbb{C}$ *closed such that* $F\supset S$ *we have*

$$E_S(F)X=X_T(F).$$

Proof. Since $\sigma(T|E_S(F)X)\subset F$, we obviously have $E_S(F)X\subset X_T(F)$.

Let us verify the inverse inclusion. Let $x\in X_T(F)$, hence $\rho_T(x)\supset\mathbb{C}\setminus F$. Let σ be a closed (compact) set such that $\sigma\bigcap F=\varnothing$. Let us proved that $E_S(\sigma)x=0$. We consider an admissible system Γ of simple Jordan curves that contains in "exterior" σ and leaves in "interior" the set F, hence $\Gamma \subset \mathbb{C} \setminus F \subset$ $\subset \rho_T(x)$. If $x(\lambda)$ is the analytic function defined on $\rho_T(x)$ such that $x = (\lambda I - T)x(\lambda)$, then $\int_\Gamma x(\lambda)\,d\lambda = 0$.

Hence we have the following relations:

$$\begin{aligned} E_S(\sigma)x &= \frac{1}{2\pi i}\int_{\|T\|+1} R(\lambda,T)E_S(\sigma)x\,d\lambda = \\ &= \frac{1}{2\pi i}\int_{\|T\|+1} R(\lambda,T \mid E_S(\sigma)X)E_S(\sigma)x\,d\lambda = \\ &= \frac{1}{2\pi i}\int_\Gamma R(\lambda,T \mid E_S(\sigma)X)E_S(\sigma)x\,d\lambda = \\ &= \frac{1}{2\pi i}\int_\Gamma E_S(\sigma)R(\lambda,T \mid E_S(\sigma)X)x\,d\lambda = \\ &= \frac{1}{2\pi i}\int_\Gamma E_S(\sigma)x(\lambda)\,d\lambda = E_S(\sigma)\frac{1}{2\pi i}\int_\Gamma x(\lambda)\,d\lambda = 0. \end{aligned}$$

The set $\mathbb{C} \setminus F$ being open we have $\mathbb{C} \setminus F = \bigcup_{n\in\mathbb{N}} \sigma_n$, with $\sigma_n \subset \mathbb{C}$ closed, $\sigma_n \subset \sigma_{n+1}$ (also σ_n can be replaced with the compact sets $\sigma_n \cap \sigma(T)$, $n \in \mathbb{N}$); consequently

$$\left(I - E_S(F)\right)x = E_S(\mathbb{C} \setminus F)x = E_S\left(\bigcup_{n\in\mathbb{N}} \sigma_n\right)x = \lim_{n\to\infty} E_S(\sigma_n)x = 0,$$

hence $x = E_S(F)x \in E_S(F)X$, whence $X_T(F) \subset E_S(F)X$.

Corollary 3.1. *Let $T \in \mathbf{B}(X)$ be an $S-$spectral operator and let E_S be one of its $S-$spectral measure. Then the map $\mathcal{E}_S$ defined by the equality*

$$\mathcal{E}_S(F) = E_S(F)X, \textit{ for } F \in \mathcal{F}_S$$

is the $S-$spectral capacity of the strongly $S-$decomposable operator T, where $\mathcal{F}_S$ is the family of all closed sets $F \subset \mathbb{C}$ such that $F \cap S = \varnothing$ or $F \supset S$.

Proof. T being $S-$spectral, then it is strongly $S-$decomposable ([7], Proposition 1.4.11.), hence it admits an $S-$spectral capacity $\mathcal{E}_S$ which is unique ([7], Theorem 2.5.5.). From the previous theorem, it follows that $E_S(F)X =$ $= X_T(F)$ if $F \supset S$ and $E_S(F)X = Y_F$ if $F \cap S = \varnothing$, where $E_S(F \cup S)X =$ $= E_S(F)X \oplus E_S(S)X = X_T(F \cup S) = Y_F \oplus X_T(S)$. In [7], Theorem 2.5.5.

and Corollary 2.5.6, it is proved that the $S-$spectral capacity of a strongly $S-$decomposable operator is given by the equalities $\mathcal{E}_S(F)=X_T(F)$, for $F\supset S$ and $\mathcal{E}_S(F)=Y_F$, for $F\cap S=\varnothing$, where the spectral maximal space Y_F is given by the relation $X_T(F\cup S)=Y_F\oplus X_T(S)$.

Remark 3.2. a) According to Theorem 3.2. and Corollary 3.1., it follows that if T is $S-$spectral and E_S is its $S-$spectral measure, then $E_S(F)X$ is a spectral maximal space of T, hence a subspace of X ultrainvariant to T, for any $F\in\mathcal{F}_S$.

b) From the assertion and the proof of the previous corollary it follows that $E_S(F)X=Y_F$, for $F\subset\mathbb{C}$ closed, $F\cap S=\varnothing$, where Y_F is the spectral maximal space of T given by the equalities $E_S(F\cup S)X=X_T(F\cup S)=Y_F\oplus X_T(S)$ and $\sigma(T\mid Y_F)\subset F$.

Theorem 3.3. *Let $T_1,T_2\in\mathbf{B}(X)$. If T_1 is $S-$spectral and T_1,T_2 are spectral equivalent, then T_2 is $S-$spectral.*

Proof. Let E_S be the $S-$spectral measure of T_1.

Since an $S-$spectral operator is strongly $S-$decomposable and the spectral equivalence transfers the property of $S-$decomposability (respectively strongly $S-$decomposability) from one operator to another (i.e. if T_1 is $S-$decomposable, respectively strongly $S-$decomposable, and $T_1\sim T_2$, then T_2 is $S-$decomposable, respectively strongly $S-$decomposable), it results that T_2 is strongly $S-$decomposable and the spectral spaces of T_1 are also the spectral spaces of T_2.

Therefore we have $X_{T_1}(F)=X_{T_2}(F)$, from $F\subset\mathbb{C}$ closed, $F\supset S$ and $Y_{1F}=Y_{2F}$, from $F\subset\mathbb{C}$ closed, $F\cap S=\varnothing$, where the spaces Y_{1F},Y_{2F} are defined by the equality $X_{T_1}(S\cup F)=X_{T_1}(S)\oplus Y_{1F}=X_{T_2}(S)\oplus Y_{2F}$. According to Theorem 3.2., it follows that $E_S(F)X=X_{T_1}(F)=X_{T_2}(F)$, for $F\supset S$ and $E_S(F)X=Y_{1F}=Y_{2F}$, for $F\cap S=\varnothing$, where $E_S(F\cup S)X=E_S(F)X\oplus E_S(S)X=$ $=Y_{1F}\oplus X_{T_1}(S)$. It results that $E_S(F)X$ is spectral maximal space of both T_1 and T_2, hence $E_S(F)X$ is invariant to T_1 and also to T_2, for $F\in\mathcal{F}_S$.

The inclusion $T_2E_S(F)X\subset E_S(F)X$ is equivalent to the equality

$$E_S(F)T_2E_S(F)=T_2E_S(F) \tag{1}$$

for any $F \subset \mathbb{C}$ closed, $F \supset S$ or $F \bigcap S = \varnothing$.

We first show that the equality (1) is verified, for any $G \subset \mathbb{C}$ open, $G \supset S$ or $G \bigcap S = \varnothing$ and then also for any Borelian set B, $B \supset S$ or $B \bigcap S = \varnothing$.

We use the fact that in a metric space every open set G is of type F_σ, i.e. $G = \bigcup_{n=1}^{\infty} F_n$, with $F_n \subset F_{n+1}$, F_n closed, $F_n \supset S$ or $F_n \bigcap S = \varnothing$. Hence, for any $G \subset \mathbb{C}$ open, $G \supset S$ or $G \bigcap S = \varnothing$ we have

$$E_S(G)T_2 E_S(G) = E_S\left(\lim_{n\to\infty} F_n\right) T_2 E_S\left(\lim_{n\to\infty} F_n\right) =$$
$$= \lim_{n\to\infty} E_S(F_n)T_2 E_S(F_n) = \lim_{n\to\infty} T_2 E_S(F_n) = T_2 E_S\left(\lim_{n\to\infty} F_n\right) =$$
$$= T_2 E_S(G),$$

Thus $E_S(G)T_2 E_S(G) = T_2 E_S(G)$.

From the relations $G = \complement F = \mathbb{C} \setminus F$, $E_S(G) = E_S(\mathbb{C}) - E_S(F) = I - E_S(F)$ and from the previous equality we have successively

$$E_S(G)T_2 E_S(G) = (I - E_S(F))\, T_2 (I - E_S(F)) = T_2 E_S(G) =$$
$$= T_2 (I - E_S(F))$$
$$T_2(I - E_S(F)) - E_S(F)T_2(I - E_S(F)) = T_2(I - E_S(F))$$
$$T_2 - T_2 E_S(F) - E_S(F)T_2 + E_S(F)T_2 E_S(F) = T_2 - T_2 E_S(F)$$

and by reducing the terms it results that

$$E_S(F)T_2 E_S(F) = E_S(F)T_2 \qquad (2)$$

for $F \subset \mathbb{C}$ closed, $F \supset S$ or $F \bigcap S = \varnothing$.

From equalities (1) and (2), for $F \subset \mathbb{C}$, $F \supset S$ or $F \bigcap S = \varnothing$ we have

$$T_2 E_S(F) = E_S(F)T_2.$$

For every $x \in X$ and $x^* \in X^* = \mathbf{B}(X, \mathbb{C})$, by using the regularity of the $S-$scalar measures $\langle E_S(\cdot)x, x^*\rangle$, $\langle T_2 E_S(\cdot)x, x^*\rangle$, $\langle E_S(\cdot)T_2 x, x^*\rangle$ we obtain

$$\langle T_2 E_S(B)x, x^*\rangle = \langle E_S(B)T_2 x, x^*\rangle$$

for any $B \in \mathcal{B}_S$ Borelian, hence

$$T_2 E_S(B) = E_S(B)T_2. \qquad (3)$$

It is known that if Y is a closed linear subspace of X invariant to $T \in \mathbf{B}(X)$, then $\|T \mid Y\| \le \|T\|$, therefore

$$\left\|\left[T_1 \mid E_S(B)X - T_2 \mid E_S(B)X\right]^{[n]}\right\|^{\frac{1}{n}} \leq \left\|(T_1 - T_2)^{[n]}\right\|^{\frac{1}{n}}.$$

But T_1 and T_2 are spectral equivalent and from the previous inequality it follows that $T_1 \mid E_S(B)X$ and $T_2 \mid E_S(B)X$ ($B \in \mathcal{B}_S$ Borelian) are spectral equivalent, thus their spectra are equal

$$\sigma(T_1 \mid E_S(B)X) = \sigma(T_2 \mid E_S(B)X),\ B \in \mathcal{B}_S.$$

Because T_1 is $S-$spectral, we have $\sigma(T_1 \mid E_S(B)X) \subset \overline{B}$, therefore

$$\sigma(T_2 \mid E_S(B)X) \subset \overline{B},\ B \in \mathcal{B}_S. \tag{4}$$

According to relations (3) and (4) it results that T_2 is $S-$spectral.

Remark 3.3. We note that the proof from the case of spectral operators ([10]) can be easily adapted to obtain the previous proof from the case of $S-$spectral operators..

Using the similar result from the case of spectral operators, we remark that the assertion of Theorem 3.3. is a consequence of [11].

Indeed, since T_2 is spectral equivalent to T_1 and $T_1 = T_1 \mid E_S(\sigma)X \oplus$ $\oplus T_1 \mid E_S(S)X$ (where $\sigma = \mathbb{C} \setminus S$) and the spectral equivalence property is transferred from operators to restrictions with respect to invariant subspaces, it follows that $T_1 \mid E_S(\sigma)X$ is spectral equivalent to $T_2 \mid E_S(\sigma)X$; hence $T_2 \mid E_S(\sigma)X$ is spectral since $T_1 \mid E_S(\sigma)X$ is spectral, $T_2 \mid E_S(S)X$ and $T_1 \mid E_S(S)X$ are spectral equivalent and their spectra are equal.

Therefore, $T_2 = T_2 \mid E_S(\sigma)X \oplus T_2 \mid E_S(S)X$ is a direct sum between a spectral operator and an operator with the spectrum in S, thus T_2 is $S-$spectral.

Theorem 3.4. *Let $T_1, T_2 \in \mathbf{B}(X)$ be two $S-$spectral operators which are spectral equivalent. Then their $S-$spectral measures E_{1S} and E_{2S} are equal, their spectral maximal spaces are equal, $X_{T_1}(F) = X_{T_2}(F)$, for $F \subset \mathbb{C}$ closed, $F \supset S$, respectively $Y_{1F} = Y_{2F}$, for $F \cap S = \emptyset$, where Y_{1F}, Y_{2F} are given by the equalities*

$$X_{T_1}(F \cup S) = Y_{1F} \oplus X_{T_1}(S),\ X_{T_2}(F \cup S) = Y_{2F} \oplus X_{T_2}(S).$$

Proof. We remark that T_1, T_2 are strongly $S-$decomposable and their spectral spaces are equal. The arguments that were used in this proof are exactly the same as in the case of spectral operators ([9], [10], [12]) and the same as in the proof of Theorem.3.3.: the equalities $E_{1S}(F)X = X_{T_1}(F) = X_{T_2}(F) = \ = E_{2S}(F)X$ (for

$F \subset \mathbb{C}$ closed, $F \supset S$), $E_{2S}(F)E_{1S}(F) = E_{1S}(F)$ (for $F \subset \mathbb{C}$ closed) and the fact that an open set G of a metric space (particularly $\mathbb{C}$) is of type F_σ (i.e. $G = \bigcup_{n=1}^{\infty} F_n$, $F_n \subset F_{n+1}$, $F_n \subset \mathbb{C}$ closed).

We find the equality $E_{2S}(G)E_{1S}(G) = E_{1S}(G)$, for $G \subset \mathbb{C}$ open, with $G \supset S$ or $G \cap S = \varnothing$ and it follows that $E_{2S}(F)E_{1S}(F) = E_{2S}(F)$, thus $E_{1S}(F) = E_{2S}(F)$, for $F \subset \mathbb{C}$ closed, $F \supset S$ or $F \cap S = \varnothing$. Finally, to obtain the equality $E_{1S}(B) = E_{2S}(B)$, for $B \in \mathfrak{B}_S$ Borelian, the regularity of the scalar measures $\langle E_{1S}(\cdot)x, x^* \rangle$, $\langle E_{2S}(\cdot)x, x^* \rangle$ is used.

We can verify the assertions of the theorem also in a simple and different manner, using similar arguments from the case of spectral operators ([10]): if T_1, $T_2 \in \mathbf{B}(X)$ are spectral operators and $T_1 \sim T_2$, then their spectral measures E_1, E_2 are equal ($E_1 = E_2$) and also their spectral spaces are equal ($X_{T_1}(F) = \, = X_{T_2}(F)$, $F \subset \mathbb{C}$ closed). Indeed, T_1, T_2 being S-spectral, then they are strongly $S-$decomposable, hence the spectral spaces $X_{T_1}(F), X_{T_2}(F)$ are equal (for $F \supset S$) and the spaces Y_{1F}, Y_{2F} are also equal (for $F \cap S = \varnothing$), where Y_{1F}, Y_{2F} are defined by the equalities $X_{T_1}(F \cup S) = Y_{1F} \oplus X_{T_1}(S) = \, = Y_{2F} \oplus X_{T_2}(S) = X_{T_2}(F \cup S)$.

According to Remark 3.1., the operators T_1, T_2 are direct sums:

$$T_1 = T_1' \oplus T_1'', \; T_2 = T_2' \oplus T_2'',$$

where

$$T_1' = T_1 \mid E_{1S}(\sigma)X, \; T_1'' = T_1 \mid E_{1S}(S)X$$

$$T_2' = T_2 \mid E_{2S}(\sigma)X, \; T_2'' = T_2 \mid E_{2S}(S)X \quad (\sigma = \sigma(T_1) \setminus S = \sigma(T_2) \setminus S)$$

and T_1' and T_2' are spectral operators with the spectral measures $E_1' = E_{1S} \mid \mathfrak{B}'_S$, $E_2' = E_{2S} \mid \mathfrak{B}'_S$, where $\mathfrak{B}'_S = \{B;\, B \in \mathfrak{B}_S, B \cap S = \varnothing\}$.

Since the restrictions to the same invariant subspace of two spectral equivalent operators are also spectral equivalent, it results that T_1' is spectral equivalent to T_2' and according to [10], Corollary 2.2.4., these operators have the same spectral measures $E_1' = E_2'$, whence $E_{1S}(B) = E_{2S}(B)$, for $B \in \mathfrak{B}_S$ Borelian, $B \cap S = \varnothing$. We use the same argument also for Borelian sets $B \in \mathfrak{B}_S$, with $B \supset S$ noting that $B = (B \setminus S) \cup S$:

$$E_{1S}(B)=E_{1S}(B\setminus S)+E_{1S}(S)=E_1{}'(B\setminus S)+E_{1S}(S)=$$
$$=E_2{}'(B\setminus S)+E_{2S}(S)=E_{2S}(B)$$

(we have $E_{1S}(S)X = X_{T_1}(S) == X_{T_2}(S)=E_{2S}(S)X$).

For the case of spectral operators, as is known, two operators T_1 and $T_2\in\mathbf{B}(X)$ are spectral equivalent if and only if their spectral measures are equal ([9], [10]). This fact seems not be true in the case of $S-$spectral operators; according to Theorems 3.3. and 3.4., the spectral equivalence transfers the property of an operator to be $S-$spectral to another operator and also implies the equality of the $S-$spectral measures; but conversely, the equality of the $S-$spectral measures does not really involve the spectral equivalence. Because we want also to fit these cases into a coherent theory, let us impose the concept of $S-$spectral equivalence (residually spectral equivalence or spectral equivalence modulo S).

Definition 3.2. Let $T_1, T_2\in\mathbf{B}(X)$ be $S-$decomposable or $S-$spectral operators with $\sigma(T_1)=\sigma(T_2)$. We say that T_1, T_2 are *$S-$spectral equivalent* (or *spectral equivalent modulo S*) if for any spectral maximal space Y of T_1 or T_2, with $\sigma(T_1\,|\,Y)\cap S=\varnothing$ or $\sigma(T_2\,|\,Y)\cap S=\varnothing$, the restrictions $T_1\,|\,Y$ and $T_2\,|\,Y$ are spectral equivalent.

Theorem 3.5. *Let $T_1, T_2\in\mathbf{B}(X)$ be $S-$spectral operators with the $S-$spectral measures E_{1S}, E_{2S}. If $E_{1S}=E_{2S}$, then T_1, T_2 are $S-$spectral equivalent. Moreover, if the "residual parts" $T_1"=T_1\,|\,X_{T_1}(S)$ and $T_2"=T_2|X_{T_2}(S)$ are spectral equivalent , then T_1 and T_2 are spectral equivalent* (*where $T_1"$ and $T_2"$ are defined in Remark* 3.1, $X_{T_1}(S)=X_{T_2}(S)=E_{1S}(S)X=E_{2S}(S)X$).

Proof. For the case of spectral operators, if two spectral operators have their spectral measures equal, then they are spectral equivalent ([10], 2.2.4.).

We use this argument for the "spectral parts" $T_1'=T_1\,|\,E_{1S}(\sigma)X$ and $T_2'=T_2\,|\,E_{2S}(\sigma)X$ (where $\sigma=\sigma(T_1)\setminus S=\sigma(T_2)\setminus S$) which, according to Remark 3.1., are spectral operators. But the spectral measures E_1', E_2' of T_1', respectively T_2' are restrictions of the $S-$spectral measures of T_1, T_2, so they are also equal, $E_1'=E_2'$, hence it follows that T_1' is spectral equivalent to T_2', whence T_1 ad T_2 are $S-$spectral equivalent.

Moreover, if $T_1"=T_1\,|\,X_{T_1}(S)$ and $T_2"=T_2|X_{T_2}(S)$ are also spectral equivalent, then $T_1=T_1'\oplus T_1"$ and $T_2=T_2'\oplus T_2"$ are spectral equivalent.

4. Conclusions

The $S-$spectral operators belong to the class of $S-$decomposable operators defined by I. Bacalu ([5], [6], [8]) and occur mainly as restrictions and quotients of decomposable (respectively, spectral, normal, self-adjoint, etc.) operators.

In this paper, certain results of the spectral theory for spectral operators are generalized to $S-$spectral operators. In the case of $S-$spectral operators, we proved that the spectral equivalence transfers the property of an operator to be $S-$spectral to another operator and also the equality of the $S-$spectral measures (similar with the case of spectral operators). But conversely, the equality of the $S-$spectral measures of two $S-$spectral operators does not involve the spectral equivalence; this equality involves only a weaker property called the $S-$spectral equivalence, notion which was defined in this paper (see Definition 3.2.).

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