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A STUDY OF A MULTI-DEGREE OF FREEDOM FRACTIONAL ORDER DAMPED OSCILLATORY SYSTEM

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The fractional calculus is a promising applied mathematical tool to different disciplines. Some dynamic systems can be precisely represented as fractional systems due to their physical properties. A multi-degree of freedom fractional damped oscillatory system is mathematically modeled by means of fractional order differential equation. In this model the damping force acting on the vibrating system is proportional to the fractional derivative of the displacement. The variable-order Caputo fractional derivative and an approximation technique are utilized to obtain the system responses. The approximation is accomplished by using a numerical discretization technique. Based on the definition of variable-order Caputo fractional derivative, the system response is investigated for different system parameters. The approximation of the system response is verified to show the efficiency of the applied techniques.

Keywords: Fractional damped oscillatory system, variable-order Caputo fractional derivative, multi-degree of freedom system, numerical discretization technique.

1. **Introduction**

The fractional calculus provides an excellent tool, via fractional derivatives, for the description of models memory and hereditary that represents the properties of various materials and processes ([1], [6]). Moreover, the fractional calculus is widely applied to many disciplines of science and engineering ([3], [4], [10], [2], [7]). In active suspension control, a fractional skyhook damping control for full-car suspension is simulated to verify the importance of proposed fractional damping in real applications ([8]).

Experiments proved that the fractional models give good approximations to the experimental data ([22], [9]). In certain applications the fractional representations show a performance improvement compared to integer representations ([5]). Some dynamic systems are accurately modeled by using fractional

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model. This is due to the behavior of included sub-systems such as viscoelastic materials or some types of dampers ([11], [12]). Some systems, such as some diffusion processes and systems with time accumulated damage, are better to be modeled as variable order fractional systems ([24]).

In some applications a variable damping force is used to suppress dynamic systems. In those systems, the damping ratio escalates depending on other parameters. As an example, we recall the Magneto-Rheological (MR) damper ([14]) and the brush disk sliding friction ([15]). Dynamic response of non-homogeneous discrete systems are simulated by using fractional order differential models. In this simulation study, the fractional calculus is used to achieve efficient and accurate model order reduction ([13]).

In this work, a hypothetical variable order fractional damped oscillating multi degree of freedom (M-dof) system is considered to be modeled. All the system elements are within single-order fractional derivatives, however in some applications, systems are modeled within the multi-order fractional derivatives ([23], [20]). The responses of the system are obtained and investigated based on system parameter's values. Due to the adaptation of initial conditions ([16]) and the derivative of a constant is zero ([17]), the variable-order Caputo fractional derivative is used to represent the fractional derivative of the system model. Together with the variable-order Caputo fractional derivative, an approximation technique is utilized to obtain the system responses. The approximation is accomplished by using a numerical discretization technique.

In this paper, some forms of the Caputo fractional derivatives are defined in section 2. The considered fractional system is described, in section 3, based on the fractional model of a single-degree of freedom system. In section 4, a M-dof system model is generated and the responses of the system are obtained by using the applied approximation techniques. In section 5, the feasibility of approximation approach is verified and the responses of the system are investigated based on some system parameters.

2. **Some preliminaries of Caputo fractional derivatives**

The fractional derivatives and integrals have two main forms. The constant order fractional derivative form and integral form are given, respectively, by:

$$ {}_aD_t^q x(t) = f(t), \tag{1a}$$

$$ {}_aD_t^{-q} f(t) = x(t), \tag{1b}$$

where q can be taken as a real or complex value. More practical formulas of the constant order fractional forms are given by variable order fractional derivatives and integrals forms, respectively, as follows:

$$ {}_aD_t^{q(t)} x(t) = g(t), \tag{2a}$$

$$ {}_aD_t^{-q(t)} g(t) = x(t), \tag{2b}$$

where $q(t)$ represents the fractional order q as a function of t. The definitions of variable order fractional derivatives and integrals forms are derived from the fractional order calculus ([24]). There are bunch of definitions such as Riemann-Liouville and Caputo fractional derivatives and integrals definitions. Based on a constant order fractional derivative, the left and right sided n-th order Caputo fractional derivatives are defined by (3) and by (4), respectively, (see [18]):

$$ {}_{a}^{C}D_{t}^{q}x(t) = \frac{1}{\Gamma(n-q)} \int_{a}^{t} (t-\sigma)^{-q-1+n} \frac{d^{n}x(\sigma)}{d\sigma^{n}} d\sigma, \qquad n-1 \leq q < n, \tag{3} $$

$$ {}_{t}^{C}D_{b}^{q}x(t) = \frac{1}{\Gamma(n-q)} \int_{t}^{b} (\sigma-t)^{-q-1+n} (-1) \frac{d^{n}x(\sigma)}{d\sigma^{n}} d\sigma, \qquad n-1 \leq q < n. \tag{4} $$

For $a = 0^{+}$, the left sided first order Caputo fractional derivative with variable order is defined by ([16]):

$$ {}_{0_{+}}^{C}D_{t}^{q(t)}x(t) = \frac{1}{\Gamma[1-q(t)]} \int_{0+}^{t} (t-\sigma)^{-q(t)} \frac{dx(\sigma)}{d\sigma} d\sigma + \frac{\left[x(0^{+}) - x(0^{-})\right] t^{-q(t)}}{\Gamma[1-q(t)]}, \tag{5} $$

where the Caputo fractional order $q(t)$ is defined by $0 \leq q(t) < 1$.

3. **The Fractional system description**

The applications of fractional calculus are exploited to describe some systems in a real manner. These systems can be found in different disciplines, such as fluid flow, control theory of dynamical systems, electro-chemistry of Corrosion and so on ([19], [26]). The freely fractional damped oscillatory dynamic system is one of these systems that can be modeled by fractional differential equation as following:

$$ m\ddot{x}(t) + c\,{}_{0}D_{t}^{q}x(t) + kx(t) = 0, \tag{6} $$

where q is a real order fractional derivative, m is the mass, k is the stiffness and the damping force represents a spring-pot ([12]) with a damping coefficient c. The model described by (6) represents a real order fractional system. In this model, a A relation between the critical damping coefficient and the order fractional derivative can be derived ([8]). Some damped systems are modeled as constant order fractional systems ([25]). However, as aforementioned in the introduction section, some systems are modeled better by means of variable order fractional systems. In some other applications, the damping coefficients are varying depending on other system parameters or system element's material behaviors ([14], [15]). Based on these behaviors of processes, systems and physical media, a hypothetical model of fractional order system can be expressed by:

$$\ddot{x}(t) + b(t)_0 D_t^{q(t)} x(t) + \omega_n^2 x(t) = 0. \tag{7}$$

Equation (7) represents the normalization of the fractional system given by (6), in which $b(t) = c(t)/m$, ω_n is the natural frequency of the fractional oscillating system and $c(t)$ is a variable damping coefficient. The model given by (7) represents a fractional oscillating single degree of freedom (S-dof) system. The considered system, in this study, is a fractional damped free vibration M-dof system (Fig.1). In this system, the responses of each element (mass) will be obtained where the damping force is proportional to fractional derivative of corresponding mass displacement. Some techniques are used to obtain the systems responses, such as Coordinate transformation, modal matrix, discretization and numerical techniques.

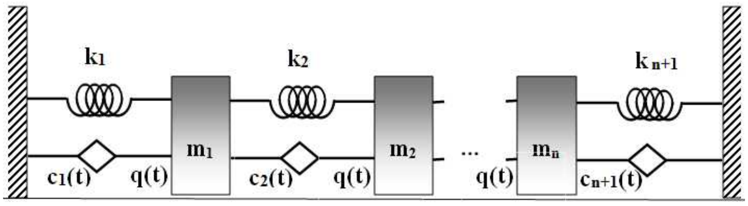

FIGURE 1. Multi-degree of freedom fractional oscillating system

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4. System Responses Approximation

4.1. System Model and Modal Analysis

The fractional damped free vibration M-dof system is analyzed based on free body diagram to obtain the system equations of motions (EOMs). From the generated EOMs the system can be modeled as follows:

$$[M]\{\ddot{x}(t)\} + [C]\left\{{}_0D_t^{q(t)} x(t)\right\} + [K]\{x(t)\} = 0, \tag{8}$$

where the matrices M, C and K, respectively, represent the mass, damping and stiffness matrices and $x(t)$ is the vector of general coordinates given by:

$$\{x(t)\} = [x_1(t)\, x_2(t) \ldots x_i(t) \ldots x_n(t)]^T, \tag{9}$$

where the general coordinate $x_i(t)$ represents the response of the mass m_i in the system and n is the number of sub-system. The modes of the system can be computed from the following Eigenvalue problem:

$$[K]\{u\} = \omega_n^2 [M]\{u\}, \tag{10}$$

where ω_n is one of the system natural frequencies and $\{u\}$ is the corresponding mode shape. The Modal matrix Φ is generated from the mode shapes of the system by:

$$[\Phi] = [\{u_1\}\{u_2\}...\{u_n\}]\,. \tag{11}$$

4.2. Coordinate Transformation

The mass, damping and stiffness matrices are transformed such that the system responses can be obtained based on the system modes. The Modal matrix is utilized to transform these matrices as follows:

$$[MT] = [\Phi]^T [M] [\Phi]\,, \tag{12a}$$

$$[KT] = [\Phi]^T [K] [\Phi]\,, \tag{12b}$$

$$[CT] = [\Phi]^T [C] [\Phi]\,, \tag{12c}$$

where the transformed mass matrix MT equals the identity matrix I, the transformed stiffness matrix KT is a diagonal matrix, in which the diagonal entries are the square of the natural frequencies ω_{n_r} of the system, the transformed damping matrix CT is a diagonal matrix, in which the diagonal entries are $2\zeta_r w_{n_r}$: $\quad r = 1, 2, ..., n$ and ζ represents a damping ratio. The system is remodeled based on the generated transformed matrices as follows:

$$[MT]\{\ddot{y}(t)\} + [CT]\left\{{}_0D_t^{q(t)} y(t)\right\} + [KT]\{y(t)\} = 0, \tag{13}$$

where the vector of principal coordinates $\{y(t)\}$ represents the transformed model responses for transformed initial conditions. The relationship between the general coordinate and the principal coordinate is given as follows:

$$\{x(t)\} = [\Phi]\{y(t)\}\,. \tag{14}$$

In order to solve the transformed model, the initial conditions of the system must be transformed to principal forms. This is because the system responses are firstly obtained based on principal coordinates and then re-transformed to actual or general coordinates so that the actual initial conditions are transformed to principal initial conditions to match the principal responses of the system. By means of the Modal matrix, the transformation of the actual initial conditions of the system positions and velocities are obtained, respectively, as follows:

$$\{y_0\} = [\Phi]^T [M]\{x_0\}\,, \tag{15a}$$

$$\{\dot{y_0}\} = [\Phi]^T [M]\{\dot{x_0}\}\,. \tag{15b}$$

The responses of the transformed fractional model given by (13) can then be approximated by using discretization and numerical techniques.

4.3. Generalization and Approximation of Transformed Model Responses

The transformed fractional model given by (13) can be rewritten as the following system of equations:

$$\begin{aligned} &\ddot{y_1}(t) + 2\zeta_1(t)\,\omega_{n_1}\,{}_0D_t^{q(t)} y_1(t) + \omega_{n_1}^2 y_1(t) = 0,\\ &\ddot{y_2}(t) + 2\zeta_2(t)\,\omega_{n_2}\,{}_0D_t^{q(t)} y_2(t) + \omega_{n_2}^2 y_2(t) = 0,\\ &\qquad\qquad\qquad\qquad \cdots\\ &\ddot{y_n}(t) + 2\zeta_n(t)\,\omega_{n_n}\,{}_0D_t^{q(t)} y_n(t) + \omega_{n_n}^2 y_n(t) = 0, \end{aligned} \tag{16}$$

where $\zeta(t)$ is a variable damping ratio. The response of each sub-equation of the system in (16) can be approximated based on its principal initial conditions. The responses approximation is accomplished by using finite differences, discretization and numerical techniques. Applying the finite difference method to the inertia terms in (16) yields

$$\ddot{y}_{i_m}(t) = \frac{y_{i_{m+1}} - 2y_{i_m} + y_{i_{m-1}}}{h^2} + O\left(h^2\right), \tag{17}$$

where $i = 1, 2, ..., n$ and h is the time increment. For simplicity, apply the approximation procedure to the $i-th$ equation of the system (16), whereas this procedure can be applied to the rest of equations, as follows substitute (17) into the $i-th$ equation in (16) to obtain the following sub-system:

$$\frac{y_{i_{m+1}} - 2y_{i_m} + y_{i_{m-1}}}{h^2} + 2\zeta_i(t_m)\,\omega_{n_i}\,{}_0D_t^{q(t_m)} y_i(t_m) + \omega_{n_i}^2 y_i(t_m) = 0. \tag{18}$$

The left sided first order Caputo fractional derivative with variable order is defined, for the $i-th$ sub-system, by ([16]):

$${}_{0_+}^{C}D_t^{q(t)} y_i(t) = \frac{1}{\Gamma[1-q(t)]}\int_{0+}^{t} (t-\sigma)^{-q(t)}\frac{dy_i(\sigma)}{d\sigma}d\sigma + \frac{\left[y_i(0^+) - y_i(0^-)\right]t^{-q(t)}}{\Gamma[1-q(t)]}. \tag{19}$$

For $y_i(0^+) = y_i(0^-)$ ([21]), a discretization technique can be applied to approximate the integral in (19) as follows:

$${}_0D_t^{q(t)} y_i(t_m) = \frac{1}{\Gamma[1-q(t_m)]}\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-q(t_m)}\frac{dy_i(\sigma)}{d\sigma}d\sigma\ . \tag{20}$$

The derivative $\frac{dy_i(\sigma)}{d\sigma}$ can be approximated at the point m by:

$$\frac{dy_{i_m}(\sigma)}{d\sigma} = \frac{y_{i_{m+1}} - y_{i_m}}{h} + O(h)\ , \tag{21}$$

where $h = t_{m+1} - t_m$ is a small increment of the variable t. Substituting the approximated derivative defined in (21) into (20) yields

$$ {}_0D_t^{q(t)} y_i\left(t_m\right) = \frac{1}{\Gamma\left[1-q\left(t_m\right)\right]} \sum_{j=0}^{m-1} \frac{y_{im+1} - y_{im}}{h} \int_{t_j}^{t_{j+1}} \left(t_m - \sigma\right)^{-q(t_m)} d\sigma \ . \quad (22) $$

Solve the integral in (22) and consider that for $j = 0, 1, 2, ..., m-1$ the following equality is valid:

$-\left[\left(t_m - t_{j+1}\right)^a - \left(t_m - t_j\right)^a\right] = h^a\left[\left(m-j\right)^a - \left(m-j-1\right)^a\right]$, where $a \in \Re$.

Opening the summation for $j = m-1$, for more details (see, e.g. [20]) and taking into account that $\left[\left(m-\left(m-1\right)\right)^{1-q(t_m)} - \left(m-\left(m-1\right)-1\right)^{1-q(t_m)}\right] = 1$ we obtain:

$$ \begin{aligned} {}_0D_t^{q(t_m)} y_i\left(t_m\right) &= \frac{h^{-q(t_m)}}{\Gamma\left[2-q\left(t_m\right)\right]} \\ &\sum_{j=0}^{m-2}\left\{\left[\left(m-j\right)^{1-q(t_m)} - \left(m-j-1\right)^{1-q(t_m)}\right]\left[y_i\left(t_{j+1}\right) - y_i\left(t_j\right)\right]\right\} \\ &+ \frac{h^{-q(t_m)}}{\left\{\Gamma\left[2-q\left(t_m\right)\right]\right\}}\left[y_i\left(t_m\right) - y_i\left(t_{m-1}\right)\right]. \end{aligned} \quad (23) $$

Substitute the approximated Caputo fractional derivative in (23) into (18) to generate the following fractional sub-system:

$$ \begin{aligned} &\frac{y_{i_{m+1}} - 2y_{i_m} + y_{i_{m-1}}}{h^2} + 2\zeta_i\left(t_m\right)\omega_{n_i}\frac{h^{-q(t_m)}}{\Gamma\left[2-q\left(t_m\right)\right]} \\ &\sum_{j=0}^{m-2}\left\{\left[\left(m-j\right)^{1-q(t_m)} - \left(m-j-1\right)^{1-q(t_m)}\right]\left[y_i\left(t_{j+1}\right) - y_i\left(t_j\right)\right]\right\} \\ &+ 2\zeta_i\left(t_m\right)\omega_{n_i}\frac{h^{-q(t_m)}}{\left\{\Gamma\left[2-q\left(t_m\right)\right]\right\}}\left[y_i\left(t_m\right) - y_1\left(t_{m-1}\right)\right]) + \omega_{n_i}^2 y_i\left(t_m\right) = 0. \end{aligned} \quad (24) $$

The response of the sub-system given in (24) can be obtained based on the principal coordinate $y_i\left(t\right)$ by applying the corresponding initial conditions obtained in (15). Concerning the rest sub-systems in (16), based on the principal coordinates, the responses can be obtained by the same procedure used for determination of $y_i\left(t\right)$. As for the M-dof system, the real responses, based on the general coordinates, are obtained by using the modal matrix as illustrated in (14).

5. Applications

The approach illustrated in the previous section is applied to two degree of freedom fractional oscillatory system with the following parameters: constant parameters: $m1 = 1,\ m2 = 1\ kg,\ k1 = 9,\ k2 = 2,\ k3 = 9\ N/m$, variable parameters: Damping coefficients and variable fractional orders. The system parameters are chosen to generate diagonal transformed matrices and mass normalization is applied to the generated eigenvectors.

5.1. Solution Verification

The responses of the two degree of freedom system are obtained by means of the introduced approximation technique. The system is subject to general initial conditions of its masses m_1 and m_2, as $x_{10} = -0.2,\ x_{20} = 0.3,\ \dot{x_{10}} = 0\ and\ \dot{x_{20}} = 0$ and constant damping coefficients $c_1 = 0.6284,\ c_2 = 0.0628,\ c_3 = 0.6284\ \ N.s^q/m$. The resulted responses are compared, as shown in (Fig. 2), with classical solutions of integer representations to verify the problem approximated solution.

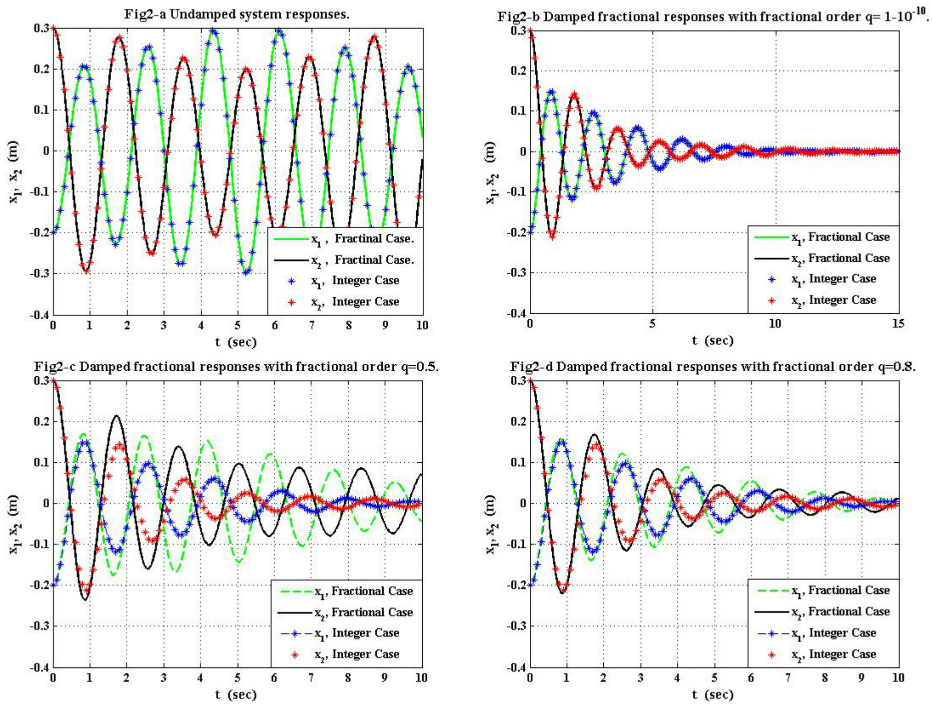

FIGURE 2. Solution verification: a- Undamped system q is close to 1, b- Damped system q is close to 1, c- Damped system $q = 0.5$, d- Damped system $q = 0.8$.

The undamped and damped systems responses are shown in (Fig. 2-a and Fig. 2-b), respectively. It's inferred from these sub-figures that for constant fractional order q be close to 1 the responses of the fractional model and integer model are almost identical. As the orders of fractional derivatives be far from one, the difference between the fractional and integer representations get larger, as shown in (Fig. 2-c and Fig. 2-d).

5.2. The Effect of System Parameters on Its Responses

The effects of the fractional order derivative of the system on its responses are illustrated in (Fig. 3). The fractional order derivative q increases, as shown in (Fig. 3-a). Consider a damped fractional system, the effect of the incremented q compared with a constant $q = 0.3$ is shown in (Fig. 3-b) where the amplitude of the responses oscillations are larger as q be closer to zero. The amplitudes become larger as q be closer to zero compared to the amplitudes of the responses for $q = 0.3$. The same effects of q on the system responses are shown in (Fig. 3-c and Fig. 3-d), where q is decreased with time.

The first and the second element responses of the fractional damped oscillatory two-dof system are shown in (Fig. 4-a and Fig. 4-b), respectively. It's shown in the figure that the responses reach steady state faster as the fractional order increases for the same damping coefficients.

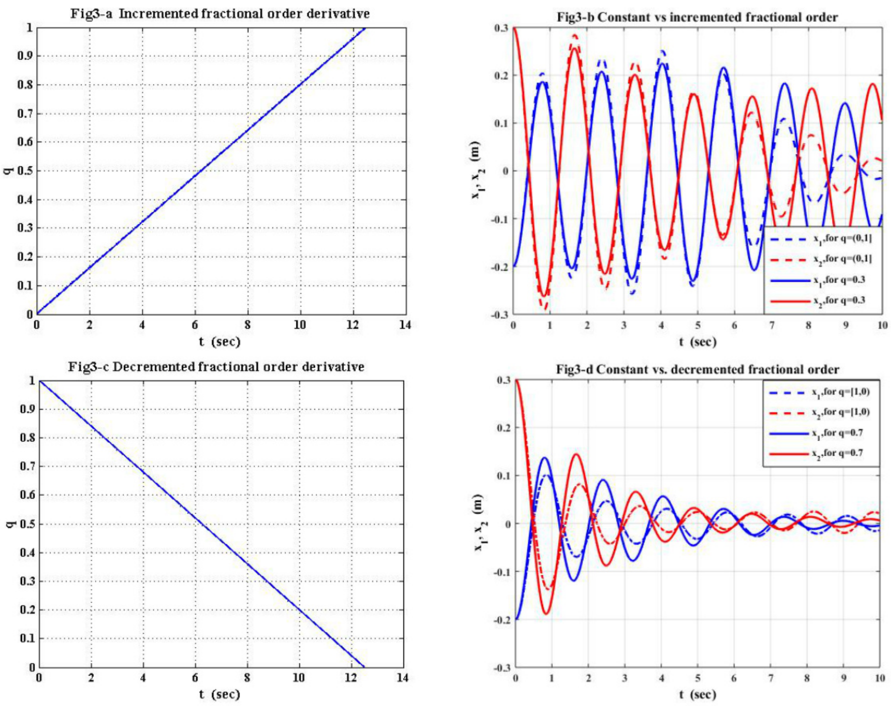

Figure 3. Damped fractional oscillating system $0 \leq q < 1$: a- Incremented fractional order derivative, b- System responses for incremented q compared to $q = 0.3$, c- decremented fractional order derivative, d- System responses for decremented q compared to $q = 0.7$.

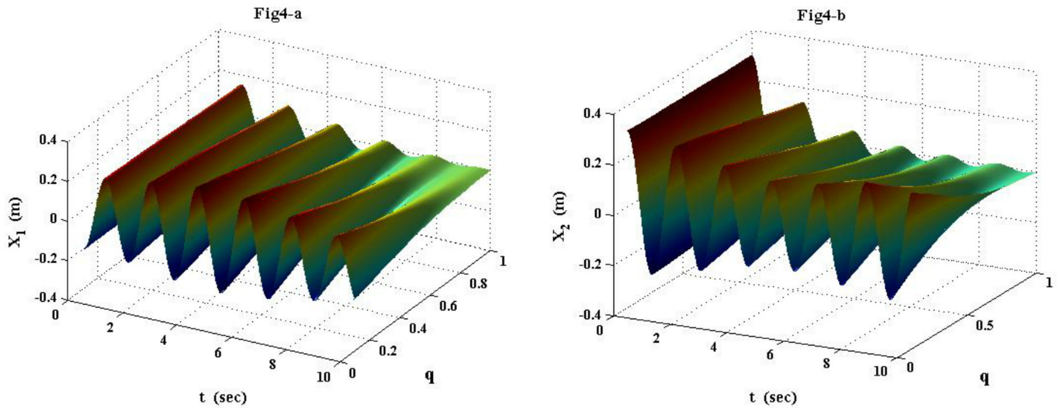

FIGURE 4. A freely fractional damped oscillating two-dof system : a- The first element responses for $0 \leq q < 1$, b- The second element responses for $0 \leq q < 1$.

The effects of the system damping ratios on the system responses are shown in (Fig. 5) where the fractional order derivative $q = 0.5$. By increasing the damping ratio the oscillations of system responses are damped faster. This looks like the effect of the damping ratio on the integer order system.

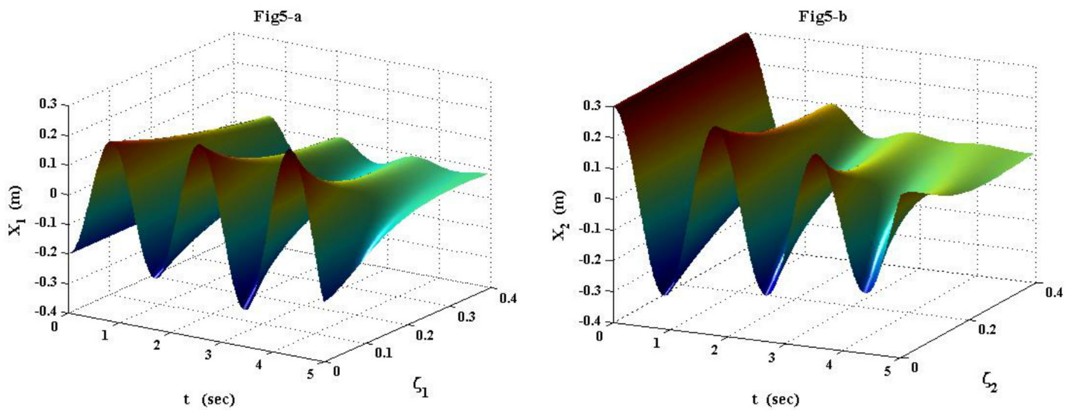

FIGURE 5. shows the effect of the damping ratio on the fractional damped two dof system responses for $q = 0.5$: a- The first element responses, b- The second element responses.

6. **Conclusion**

In this work a fractional damped oscillating M-dof model is studied. The variable-order Caputo fractional derivative and numerical discretization techniques are considered to approximate the system responses. The feasibility of the introduced approximation is verified, as shown in (Fig. 2), by comparing the damped and undamped fractional system responses, for q be close to one, with the integer case. The effects of system parameters on the system responses are investigated. The effect of the variable order derivative q on the system responses shows the following:

- The amplitude of the responses oscillations, as illustrated in (Fig. 3), are larger as q be closer to zero compared to the system responses for $q = 0.3$ and $q = 0.7$.
- The responses reach steady state faster as the fractional order q increases for the same damping coefficients, as illustrated in (Fig. 4).

For specific fractional order derivative $q = 0.5$ the effects of the system damping ratios are investigated, as shown in (Fig. 5). The effects of the damping forces show that as the damping ratios increase the system responses are damped faster and the overshoots become lower. These effects which are valid for $0 \leq q < 1$ look like the effect of the damping ratio on the integer order system.

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