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APPLICATION OF LAGRANGE EQUATIONS FOR CALCULUS OF INTERNAL FORCES IN A MECHANISM

Thien Van NGUYEN[1], Roxana Alexandra PETRE[2], Ion STROE[3]

One of the great strengths of Lagrange procedure is to deal easily kinematic problems with any coordinate system by using a set of generalized coordinates. In addition, the dynamic analysis of motion of a system subjected to known external forces also is more convenient by without considering constrained forces. Based on the Lagrange equations, this paper presents a method to directly determine internal forces in a rigid body of a mechanism.

Keywords: Dynamics, internal force, slider-crank mechanism, constraints

1. Introduction

A major difficulty in finding the solution of any problem in mechanics is the selection of that coordinate system which will leave the equations of motion in a form most amenable to further treatment.

For a large class of mechanical systems, the Lagrange equations provide a unique and sufficiently simple method for formulating equations of motion that is independent of the complexity of the actual system. The chief advantage of the Lagrange equations is that their number is equal to the number of degrees-of-freedom and is independent of the number of points and bodies in the system. For example, a mechanism consists of many components and has just one or two degrees of freedom. Consequently, the study of its motion requires the setting up of only one or two Lagrange equations. In addition, under ideal conditions, all unknown constraints are automatically excluded from the Lagrange equations. For these reasons the Lagrange equations are widely used in the solution of many problems in mechanics, in particular, problems dealt with the dynamic analysis.

As we know, determining of internal forces, constraint force analysis are the important steps in dynamic analysis, which is the base of structure design of mechanism. In fact, the calculus of internal forces for a static system of rigid bodies is quite familiar in the field of material resistance. Firstly, the constrained forces need to be computed. Based on that, the internal forces such as: axial force,

[1] PhD Student, Department of Mechanics, University POLITEHNICA of Bucharest, Romania, email: bangden33468@gmail.com
[2] Department of Mechanics, University POLITEHNICA of Bucharest, Romania
[3] Department of Mechanics, University POLITEHNICA of Bucharest, Romania

shearing force and bending moment at specified points in the rigid body, are determined with the help of classical method, namely the cutting plane method. However, the calculus of constrained forces in a mechanism composing of complicated structure and possessing a large number of degrees of freedom, is extremely difficult. Consequently, the determining of internal forces will be faced unavoidable difficulties.

In recent years, the problems related to dynamic analysis attracted the attention of many researchers and some have got valuable results in this field:

Lu [2] uses the virtual work method to determine, in spatial parallel structures, the generalized forces of actuators and relates them to the real forces that they exert. Geike and Mc Phee [3] proposed a general approach which could determine the inverse dynamics solutions for a planar 3-RRR parallel manipulator and spatial 6-DOF parallel mechanism. Jiang, Li and Wang [10] using Newton Euler method and D'Alembert principle established the force analysis equations, and also put forward the dynamic analysis model of a parallel mechanism based on the deformation compatibility method. Zhi and Wang [11] improved and applied the solution method of reciprocal screw system to solution procedure of the constraint forces.

By using a set of *generalized coordinates* that are consistent with the constraint relations, we can formulate the equations of motion and calculate the internal forces in the multi-body system without considering constraint forces. If an internal force has to be found, a supplementary mobility related to it is considered in the system, and the corresponding internal force for new mobility is found for null values of mobility as well as for its first and second derivatives.

The slider-crank mechanism is one of the most commonly used machine subsystem in mechanical system design. It is employed as the principal element of internal combustion engines, compressors, fly-ball governors, stamping machines, and many other machines. Therefore, the slider-crank mechanism is considered as a simple model for calculating internal forces in the connecting rod of the system at an instant moment of time to illustrate for that method.

2. Lagrange Equations

2.1. Equations of motion of the system

We introduce a general notation for the relationship between h Cartesian variables of position x_i and their description in generalized coordinates (for some systems, the number of generalized coordinates is lager than the number of degrees of freedom and this is accounted for by introducing constraints on the system). In the general case, each x_i could be dependent upon every q_k.

$$x_i = x_i\left(q_1, q_2 ... q_k ..., q_h\right) \tag{1}$$

When constraints are expressed by functions of coordinates, the motion of the systems can be studied with Lagrange equations for holonomic systems with dependent variables, whereas other conditions of constraint are expressed by velocities, the motion is described with Lagrange equations for non-holonomic systems.

For a non-holonomic system, the Lagrange equations corresponding to a system of h generalized coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right)-\frac{\partial E}{\partial q_k}=Q_k+\sum_{i=1}^{p}\lambda_i a_{ik}, \quad (k=1,2,...,h) \tag{2}$$

are completed with the constraints

$$\sum_{k=1}^{h} a_{ik}\dot{q}_k+b_i=0, \quad (i=1,2,...,p), \tag{3}$$

where E is the kinetic energy expressed with respect to an inertial reference frame, conventionally considered as fixed, and

$$Q_k=\frac{\delta W_k}{\delta q_k} \tag{4}$$

are the generalized forces, while δW_k is the virtual work produced by the forces acting upon the system, corresponding to the virtual displacement δq_k.

By solving a system of (2), of h equation, and (3), of p equations, coordinates q_k and Lagrange multipliers λ_i will be found.

From (2), the equations for the holonomic system can be obtained by replacing functions a_{ik}. In the case of a holonomic system, constraints are of the form:

$$\Phi_i\left(q_1,...,q_h,t\right)=0, \quad (i=1,2,...,p) \tag{5}$$

From the above formula, the following differential form is obtained:

$$\sum_{k=1}^{h}\frac{\partial \Phi_i}{\partial q_k}\dot{q}_k+b_i=0, \quad (i=1,2,...,p) \tag{6}$$

By comparing relations (6) and (3), it follows

$$a_{ik}=\frac{\partial \Phi_i}{\partial q_k} \tag{7}$$

Then the equations (2) become:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right)-\frac{\partial E}{\partial q_k}=Q_k+\frac{\partial}{\partial q_k}\sum_{i=1}^{p}\lambda_i\Phi_i, \quad (k=1,2,...,h) \tag{8}$$

Let define the analytical function

$$U_{\Phi} = \sum_{i=1}^{p} \lambda_i \Phi_i \tag{9}$$

then equations (8) can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = Q_k + \frac{\partial U_{\Phi}}{\partial q_k}\ , \quad \left(k = 1, 2, ..., h\right) \tag{10}$$

Starting from these h differential equations with using p relations of constraints, we determine just the generalized coordinates q_k and the Lagrange multipliers.

2.2. Calculus of internal forces

For a mechanical system with h degrees of freedom represented by independent generalized coordinates q_k (k=1,2…,h), the Lagrange equations are expressed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = \frac{\partial U}{\partial q_k} + Q_k^{*}\ , \quad (k = 1, ..., h) \tag{11}$$

An internal force Q_{h+1}, as the new generalized force, can be found if a new fictitious mobility according to the internal force is considered. Then the mechanical system is considered as the one with $h+1$ degrees of freedom. The equation for the new mobility is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_{h+1}}\right) - \frac{\partial E}{\partial q_{h+1}} = \frac{\partial U}{\partial q_{h+1}} + Q_{h+1}\,. \tag{12}$$

Considering again the mechanism, the internal force $\Re_{h+1}$ is easily obtained from (12) in the following form

$$\Re_{h+1} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_{h+1}}\right) - \frac{\partial E}{\partial q_{h+1}} - \frac{\partial U}{\partial q_{h+1}}\right]_{\substack{q_{h+1}=0 \\ \dot{q}_{h+1}=0 \\ \ddot{q}_{h+1}=0}} \tag{13}$$

3. Applying Lagrange equations in calculus internal forces in a mechanism

As an example, the one degree of freedom system of slider-crank mechanism is considered (Fig.1). The slider-crank mechanism consists of the crank *1* characterized of the length OA= r, mass m_1; the connecting rod *2* characterized by the length AB= l, mass m_2 and the slider *3* characterized by mass m_3. The crank OA rotates by the active torque M_o.

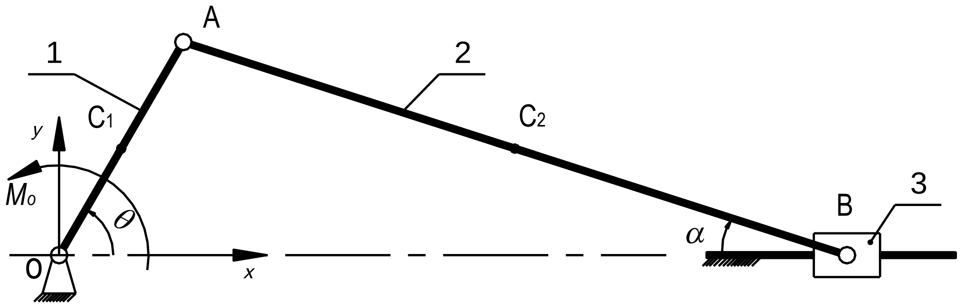

Fig. 1. Slider-crank mechanism

This mechanism has one degree of freedom. Therefore, by applying (10) it is easy to find out the differential equation governing the motion

$$\left(\begin{array}{l}\dfrac{m_1}{3}+\left(\dfrac{m_2.l^2}{3}+m_3.r^2.\sin^2\theta\right)\dfrac{\cos^2\theta}{l^2-r^2.\sin^2\theta}+\\ +\left(m_2+m_3\right).\sin^2\theta+\dfrac{(m_2+2m_3).r.\sin^2\theta.\cos\theta}{\sqrt{l^2-r^2.\sin^2\theta}}\end{array}\right)r^2\ddot{\theta}+$$

$$+\left(\begin{array}{l}\left(\dfrac{3m_2}{4}+m_3\right).\sin\theta.\cos\theta-\left(\dfrac{m_2}{2}+m_3\right).\dfrac{r.\sin^3\theta}{\sqrt{l^2-r^2.\sin^2\theta}}+\\ +\left(m_2+2m_3\right).\dfrac{r.\sin\theta.\cos^2\theta}{\sqrt{l^2-r^2.\sin^2\theta}}+\dfrac{m_2.r^2.l^2.\sin\theta.\cos^3\theta}{12(l^2-r^2.\sin^2\theta)^2}+\\ +\left(\dfrac{m_2}{4}+m_3\right)\dfrac{r^2.\sin\theta.\cos\theta.(\cos^2\theta-\sin^2\theta)}{l^2-r^2.\sin^2\theta}+\\ +\left(\dfrac{m_2}{2}+m_3\right).\dfrac{r^3.\sin^3\theta.\cos^2\theta}{\left(l^2-r^2.\sin^2\theta\right)\sqrt{l^2-r^2.\sin^2\theta}}-\dfrac{m_2.l^2.\sin\theta.\cos\theta}{12(l^2-r^2.\sin^2\theta)}\end{array}\right)r^2\dot{\theta}^2 \qquad (14)$$

$$=M-\frac{\left(m_1+m_2\right).g.r.\cos\theta}{2}$$

Supposing the supplementary displacement corresponding to the constrained force N_B is "v" as shown in Fig.2. Thus, for determining the constrained force at the point B at an instant moment of time, the generalized coordinates of the slider-crank mechanism are represented by $q_1=\theta;\ q_2=v.$

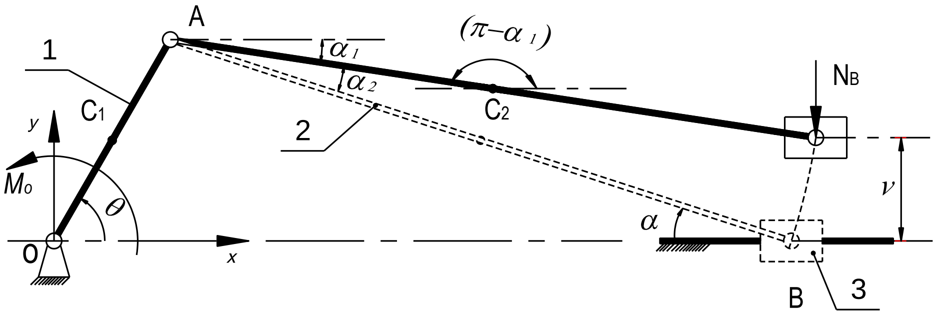

Fig. 2. Virtual supplemental displacement corresponding to the constrained force

The constrained force will be found by using Lagrange equation

$$N_B = -\Re_{h+1} = -\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{v}}\right) - \frac{\partial E}{\partial v} - \frac{\partial U}{\partial v}\right]_{\substack{v=0\\ \dot{v}=0\\ \ddot{v}=0}} \tag{15}$$

The kinetic energy will be expressed as following:

$$E = \frac{1}{2}\{\omega_1\}^T .J_o.\{\omega_1\} + \frac{1}{2}\{\dot{r}_{C2}\}^T .m_2.\{\dot{r}_{C2}\} + \\ + \frac{1}{2}\{\omega_2\}^T .J_{C2}.\{\omega_2\} + \frac{1}{2}\{\dot{r}_{C3}\}^T .m_3.\{\dot{r}_{C3}\} , \tag{16}$$

where $\{\omega_2\}$,being the angular velocity of the AB rod, is expressed by

$$\{\omega_2\} = -\{\dot{\alpha}_1\} = \left[0,\ \ 0,\ \ \frac{(l^2 - r^2.\sin^2\theta)\dot{v} + r^2 v\dot{\theta}.\sin\theta.\cos\theta}{(l^2 - r^2.\sin^2\theta)\sqrt{l^2 - r^2.\sin^2\theta}} - \frac{r\dot{\theta}.\cos\theta}{\sqrt{l^2 - r^2.\sin^2\theta}}\right]^T , \tag{17}$$

and $\{r_{C2}\}$, $\{r_{C3}\}$, which are respectively position vectors of mass centers of the two parts m_2 and slider m_3.

The force function U has the expression

$$U = -\frac{m_1 g r.\sin\theta}{2} - m_2 g\left(\frac{r.\sin\theta}{2} + \frac{r v^2.\sin\theta}{4(l^2 - r^2.\sin^2\theta)} + \frac{v}{2}\right) - m_3 g v\ . \tag{18}$$

After taking the partial derivatives and derivatives with respect to time for the terms related to the Lagrange equations, then applying (15), the constrained force is obtained

$$N_B = \frac{\left(\begin{pmatrix} 3m_2r^6.\sin^6\theta + 6m_3r^6.\sin^6\theta + 3m_2r^2l^4.\sin^2\theta + \\ +6m_3r^2l^4.\sin^2\theta - 6m_2r^4l^2.\sin^4\theta - 12m_3r^4l^2.\sin^4\theta \end{pmatrix}\ddot{\theta} + \\ + \begin{pmatrix} 3m_2r^6.\sin^5\theta.\cos\theta + 6m_3r^6.\sin^5\theta.\cos\theta + \\ -6m_2r^4l^2.\sin^3\theta.\cos\theta - 12m_3r^4l^2.\sin^3\theta.\cos\theta + \\ +3m_2r^2l^4.\sin\theta.\cos\theta + 6m_3r^2l^4.\sin\theta.\cos\theta \end{pmatrix}\dot{\theta}^2\right)}{6(l^2 - r^2.\sin^2\theta)\sqrt{(l^2 - r^2.\sin^2\theta)}} +$$

$$+ \frac{\left(\begin{pmatrix} 3m_2r^5.\sin^4\theta + 6m_3r^5.\sin^4\theta + m_2rl^4.\cos\theta + \\ -4m_2r^3l^2.\sin^2\theta.\cos\theta - 6m_3r^3l^2.\sin^2\theta.\cos\theta \end{pmatrix}\ddot{\theta} + \\ + \begin{pmatrix} -3m_2r^5.\sin^5\theta - 6m_3r^5.\sin^5\theta + 2m_2r^3l^2.\sin\theta + \\ +6m_3r^3l^2.\sin\theta - m_2rl^4.\sin\theta + \\ -4m_2r^3l^2.\sin\theta.\cos^2\theta - 12m_3r^3l^2.\sin\theta.\cos^2\theta \end{pmatrix}\dot{\theta}^2\right)}{6(l^2 - r^2.\sin^2\theta)} - \frac{m_2g}{2} - m_3g \cdot \quad (19)$$

After obtaining the result of constrained force N_B, replacing the constraint at the end B by the action of the force N_B, then the mechanism is considered as an open chain.

For determining the N axial force in the connecting AB, supposing the supplementary displacement corresponding to the axial force is "u" as shown in Fig.3. Thus, the generalized coordinates of the slider-crank mechanism are represented by $q_1 = \theta;\ q_2 = u.$

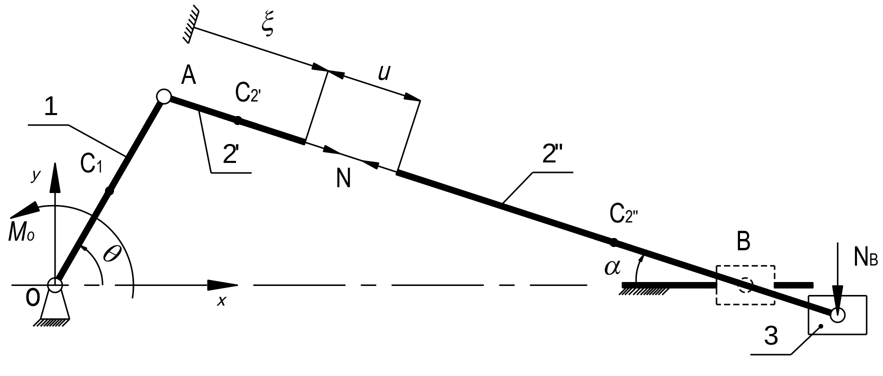

Fig. 3. Virtual supplemental displacement corresponding to the axial force

Likewise, supposing the supplementary displacement corresponding to the shearing force F is "s" as shown in Fig.4. Thus, the generalized coordinates of the slider-crank mechanism are represented by $q_1 = \theta;\ q_2 = s.$

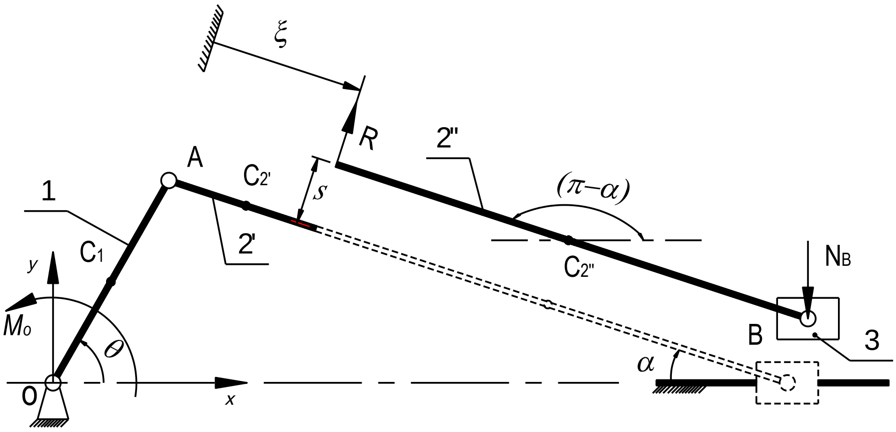

Fig. 4. Virtual supplemental displacement corresponding to the shearing force

And supposing the supplementary displacement corresponding to the bending moment M is “φ” as shown in Fig.5. Thus, the generalized coordinates of the slider-crank mechanism are represented by $q_1 = \theta;\ q_2 = \varphi.$

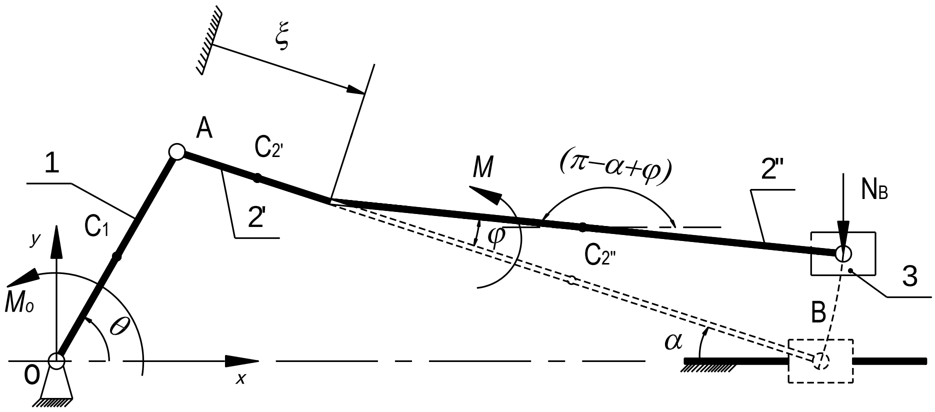

Fig. 5- Virtual supplemental displacement corresponding to the bending moment

The axial force, shearing force and bending moment will be found by using the Lagrange equation

$$\Re_{h+1} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_{h+1}} \right) - \frac{\partial E}{\partial q_{h+1}} - \frac{\partial U}{\partial q_{h+1}} - \frac{\delta L_{q_{h+1}}(\vec{N}_B)}{\delta q_{h+1}} \right]_{\substack{q_{h+1}=0 \\ \dot{q}_{h+1}=0 \\ \ddot{q}_{h+1}=0}}, \qquad (20)$$

After taking the partial derivatives and derivatives with respect to time for the terms related to the Lagrange equations for each case, then applying (20), the axial force N is obtained

$$N=\frac{\begin{pmatrix}\begin{pmatrix}6m_2r^6\xi.\sin^5\theta.\cos\theta-5\mathrm{m}_2r^2l^5.\sin\theta.\cos\theta-6m_3r^2l^5\sin\theta.\cos\theta+\\ -3m_2r^6l.\sin^5\theta.\cos\theta+8m_2r^4l^3\sin^3\theta.\cos\theta+6m_3r^4l^3.\sin^3\theta.\cos\theta+\\ -12m_2r^4l^2\xi.\sin^3\theta.\cos\theta+6m_2r^2l^4\xi.\sin\theta.\cos\theta\end{pmatrix}\ddot{\theta}+\\ +\begin{pmatrix}-6m_2r^6\xi.\sin^6\theta+3m_2r^6l.\sin^6\theta-3m_2r^2l^5.\cos^2\theta-6m_3r^2l^5.\cos^2\theta+\\ 2m_2r^4l^3.\sin^2\theta+5m_2r^2l^5.\sin^2\theta+6m_3r^4l^3.\sin^2\theta+6m_3r^2l^5.\sin^2\theta+\\ -10m_2r^4l^3\sin^4\theta-12m_3r^4l^3.\sin^4\theta+3m_2r^2l^3\xi^2.\cos^2\theta+\\ -m_2r^4l^3.\sin^2\theta.\cos^2\theta-6m_3r^4l^3.\sin^2\theta.\cos^2\theta-6m_2r^2l^4\xi.\sin^2\theta+\\ +12m_2r^4l^2\xi.\sin^4\theta-3m_2r^4l\xi^2.\sin^2\theta.\cos^2\theta\end{pmatrix}\dot{\theta}^2+\\ +6m_2gr^3l^3.\sin^3\theta-3m_2grl^5.\sin\theta+6m_2gr^5\xi.\sin^5\theta+\\ -3m_2gr^5l.\sin^5\theta-12m_2gr^3l^2\xi.\sin^3\theta+6m_2grl^4.\sin\theta\end{pmatrix}}{6.l^2(l^2-r^2.\sin^2\theta)^2}+$$

$$+\frac{\begin{pmatrix}\begin{pmatrix}-12m_2r^5l^3.\sin^5\theta+12m_3r^3l^5.\sin^3\theta+15m_2r^3l^5.\sin^3\theta+\\ -6m_3r^5l^3.\sin^5\theta-6m_2rl^7.\sin\theta-6m_3rl^7.\sin\theta+\\ -6m_2r^7\xi.\sin^7\theta+3m_2r^7l.\sin^7\theta-18m_2r^3l^4\xi.\sin^3\theta+\\ 18m_2r^5l^2\xi.\sin^5\theta+6m_2rl^6\xi.\sin\theta\end{pmatrix}\ddot{\theta}+\\ +\begin{pmatrix}-6m_2rl^7.\cos\theta-6m_3rl^7.\cos\theta+6m_2rl^6\xi.\cos\theta+15m_2r^3l^5.\sin^2\theta.\cos\theta+\\ +12m_3r^3l^5.\sin^2\theta.\cos\theta-12m_2r^5l^3.\sin^4\theta.\cos\theta-6m_3r^5l^3.\sin^4\theta.\cos\theta+\\ -6m_2r^7\xi.\sin^6\theta.\cos\theta+3m_2r^7l.\sin^6\theta.\cos\theta-18m_2r^3l^4\xi.\sin^2\theta.\cos\theta+\\ +18m_2r^5l^2\xi.\sin^4\theta.\cos\theta\end{pmatrix}\dot{\theta}^2\end{pmatrix}}{6.l^2(l^2-r^2.\sin^2\theta)^2.\sqrt{l^2-r^2.\sin^2\theta}}$$

(21)

Similarly, final results can be obtained for the shearing force F and bending moment M.

In the inverse dynamics, we suppose that the history of rotational motion of the crank is given by the following function:

$$\theta=\Omega_o t+\frac{\varepsilon_o}{2}t^2, \qquad (22)$$

where $\Omega_o = 25\pi\ (rad/s)$, $\varepsilon_o = \frac{\pi}{100}\ (rad/s^2)$. And the variation of the angle θ with respect to time is shown in the Fig. 6.

For the simulation purpose let's consider the slider-crank mechanism with the following characteristics: m_1=0.1(kg), r=0.1 (m), m_2=0.1 (kg), l=0.2 (m), m_3=0.1 (kg)

Using MATLAB software, a program was developed to solve the inverse dynamics of the slider-crank mechanism. The variations of the active torque, constrained force versus time are shown respectively in Fig.7 and Fig.8, the variation of the internal forces with respect to ratio ξ/l are shown in Fig.9, Fig.11, Fig.13 and the variations of the internal forces at three specified positions versus time are shown in Fig.10, Fig.12, Fig.14.

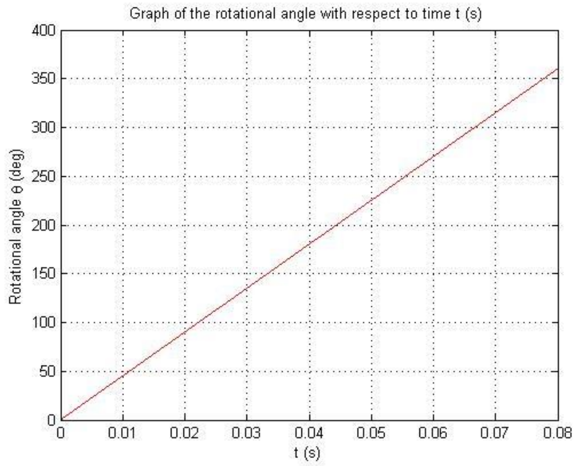

Fig. 6. The rotational angle θ.

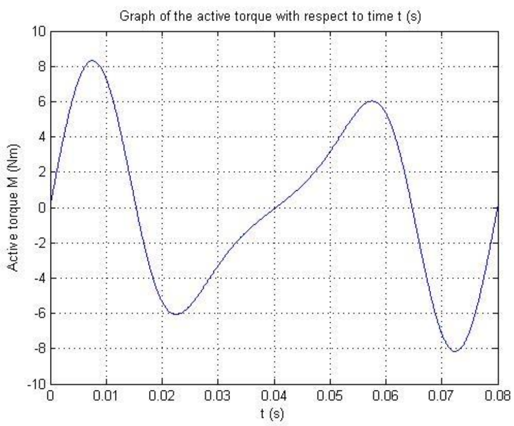

Fig. 7. The active torque of the crank M.

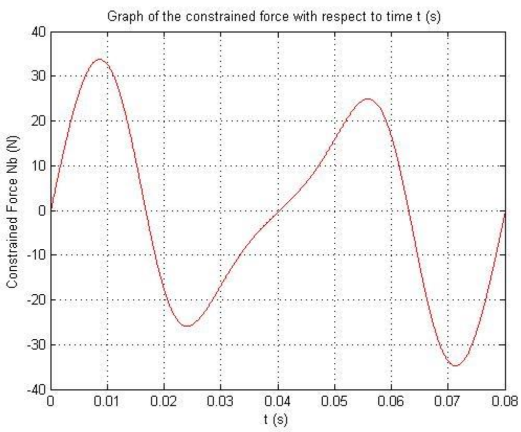

Fig. 8. The constrained force at the end B

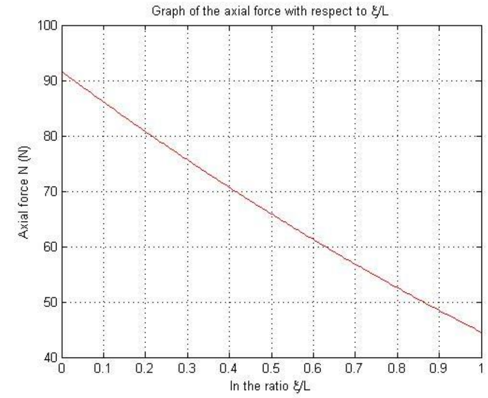

Fig. 9. The axial force with respect to "ξ/l " at the instant time t=0.03(s).

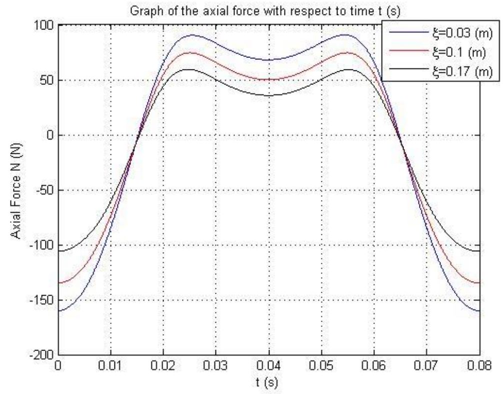

Fig. 10. The axial forces at the three specified positions versus time.

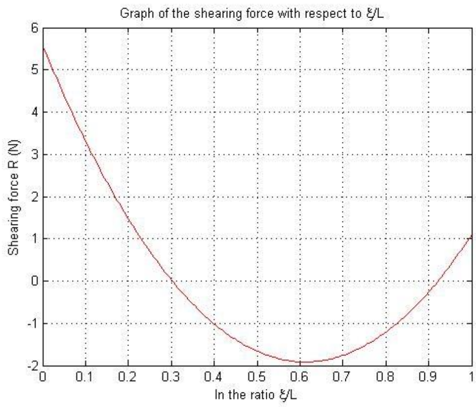

Fig. 11. The shearing force with respect to "ξ/l " at the instant time t=0.03(s).

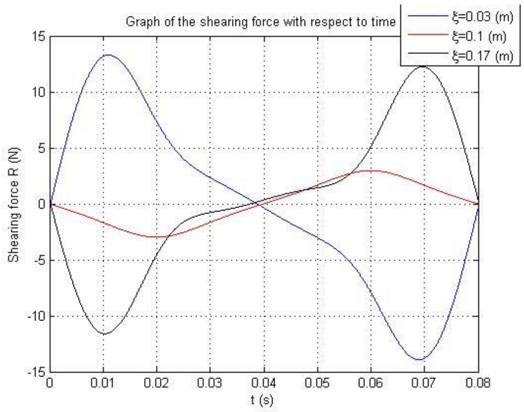

Fig. 12. The shearing forces at the three specified positions versus time.

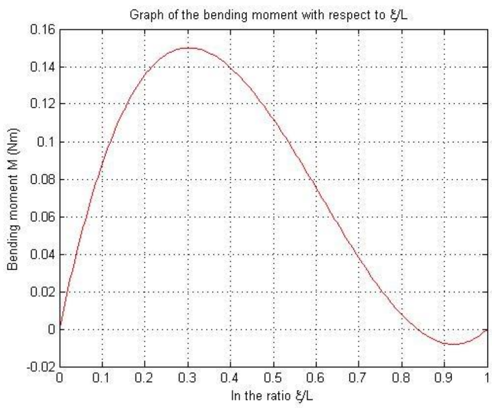

Fig. 13. The bending moment with respect to "ξ/l" at the instant time t=0.03(s).

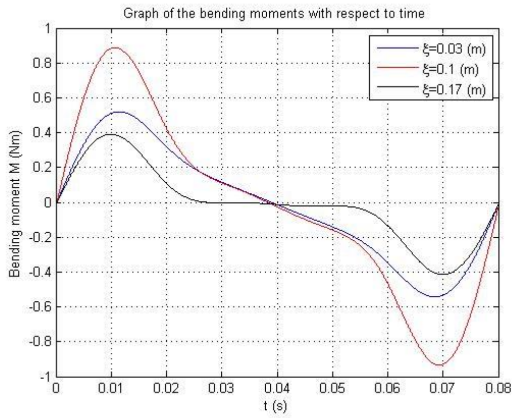

Fig. 14. The bending moments at the three specified positions versus time.

5. Conclusions

With the Lagrange equations, the position, velocity and acceleration of each element of the mechanism in real-time revolution can be released from the differential equations of motion in the kinetic analysis. In addition, the active torque consistent with the given motional law has been determined by taking into account the masses and forces of inertia introduced by the links of the mechanism in the model dynamics. Besides, after calculating the constrained force and then replacing for the constraint corresponding to it, we can consider a closed chain as an open chain being used very popular in the robotic arm. The most important fact is that the present paper introduced a convenient method for determining directly internal forces in an arbitrary rigid body in a multibody system. The results are shown above as a demonstration for this approach since the correspondence with values between the internal forces.

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