NONDIFFERENTIABLE MINIMAX PROGRAMMING PROBLEMS UNDER HIGHER-ORDER CONVEXITY

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The aim of this paper is to study a nondifferentiable minimax programming problem with square root terms in the objective functions and establish sufficient optimality conditions from the standpoint of the higher-order convexity assumptions. These optimality conditions are illustrated by a non-trivial example. Furthermore, weak, strong, and strict converse duality theorems are derived for two dual model categories.

Keywords: Minimax programming, nondifferentiable programming, strong convexity of order m, sufficient optimality conditions, duality.

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1. Introduction

In spite of optimization problems that have been present in mathematics as the earliest times, optimization has been established as an independent field only in relatively recent times. The idea of a minimax programming problem, without a doubt, plays a significant role in all parts of programming in mathematics including the duality theorem and optimality conditions. Some blatant and significant outcomes of minimax programming problems were considered in books Danskin [6] and Demyanov and Malozemov [7]. Optimality and duality constitute a vital piece in the investigation of mathematical programming in the sense that these lay down the foundation of the algorithm for a solution of an optimization problem.

In recent years, there has been growing interest in minimax mathematical programming (see, for example, [1, 2, 3, 4, 8, 9, 10, 13, 14, 17, 18, 20, 21, 22, 23]). For a generalized minimax programming problem, Schmitendorf [18] first presented necessary and sufficient optimality conditions, which Tanimoto [20] used to formulate a dual problem and to discuss about the duality outcomes. Bector and Bhatia [4] and Weir [21] also employed the aforesaid method to construct several dual models and derived weak and strong duality theorems under pseudoconvex and quasiconvex functions. Under generalized invexity hypotheses, Zalmai [23] established sufficient optimality conditions and duality theorems, and he introduced necessary optimality conditions for a class of minimax programming problems in Banach space. Later, Bector et al. [5] developed duality results for a class of minimax programming problems with V-invexity-type assumptions on the goal and constraint functions.

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The concept of convexity and generalized convexity plays a dominant role in several sides of mathematical programming and other related fields. Recently, various generalized convexities have been introduced, and one proof that can be applied to larger classes of optimization problems to extend optimality conditions and duality conclusions for convex programs. A significant generalization of the convex function is the strong convexity of order m which was introduced by Karamardian [12]. Such generalization was introduced for a differentiable function by Lin and Fukushima [16] and thereafter utilized by several authors to get significant results. The concept of strong convexity is a fundamental tool for designing and analyzing a wide range of learning algorithms. Suneja et al. [19] proved optimality and duality results for nonsmooth vector optimization problems under generalized higher-order strongly convex functions. Later, Jayswal et al. [11] concentrated their study to investigated sufficient and general Mond-Weir type duality results for nonlinear multiobjective programming problems in which the function involved are semidifferentiable under higher order semilocally strong convexity.

In this paper, we consider a class of optimization problems with square root terms in the objective functions governed by the higher-order strong convex functions. The results obtained here include the results formulated in the previous mentioned studies. The presence of continuous differentiable of functions of minimax optimization problems with square root terms in the objective functions represent the main element of total novelty in the specialized literature. Thereafter, we formulate and prove sufficient optimality conditions associated with the considered non-differentiable minimax optimization problems, under higher-order strong convex hypotheses of the involved functions. Along these lines, we apply the optimality conditions to define dual models and prove weak, strong and strict converse duality theorems using higher-order strong convex assumptions.

The organization of this paper is as follows: Section 2 contains some preliminary notions and basic definitions needed in the sequel. In Section 3, we have derived the sufficient optimality conditions for a class of nondifferentiable minimax programming problems which is illustrated by a non-trivial example. In Sections 4-5, we study two types of dual models namely Wolfe type dual and general Mond-Weir type dual, respectively, and establish weak, strong and strict converse duality results for each of them. Finally, Section 6 provides a conclusion and future developments.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+$ be its non-negative orthant and X be a non-empty open subset of $\mathbb{R}^n$.

The problem to be considered in the present analysis is the following nondifferentiable minimax programming problem :

(NP) min $\sup_{y\in Y} f(x,y) + (x^T Ax)^{\frac{1}{2}}$

subject to $g(x) \leq 0,\ x \in X$,

where Y is a compact subset of $\mathbb{R}^l; f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ are continuous differentiable functions and A is a positive semidefinite $n \times n$ symmetric matrix. If $A = 0$, then the problem (NP) is a differentiable minimax programming problem introduced by Schmitendorf [18].

Let $\Omega = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ denote the set of all feasible solutions of (NP). For each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$ we define

$J(x) = \{j \in K : g_j(x) = 0\}$, where $K = \{1, 2, ..., k\}$,

$$Y(x) = \{y \in Y : f(x, y) + (x^T Ax)^{\frac{1}{2}} = \sup_{z \in Y} f(x, z) + (x^T Ax)^{\frac{1}{2}}\}.$$

$$M(x) = \{(p, \lambda, \tilde{y}) \in \mathbb{N} \times \mathbb{R}^p_+ \times \mathbb{R}^{lp} : 1 \leq p \leq n+1, \lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in \mathbb{R}^p_+ \text{ with } \sum_{i=1}^{p} \lambda_i = 1 \text{ and } \tilde{y} = (\bar{y_1}, \bar{y_2}, ..., \bar{y_p}) \text{ with } \bar{y_i} \in Y(x),\ i = 1, 2, ..., p\}.$$

Now, we begin with the definitions of strong convexity given by Lin and Fukushima [16].

Definition 2.1. *(Lin and Fukushima* [16]*). Let $m \geq 1$ be an integer number. A differentiable function $h : X \to \mathbb{R}$ is said to be (strictly) strongly convex of order m at $x^* \in X$, if there exists a positive constant c such that*

$$h(x) - h(x^*)(>) \geq (x - x^*)^T \nabla h(x^*) + c\|x - x^*\|^m, \forall\ x \in X,$$

where $\| \cdot \|$ denotes any norm of X. If the function h is (strictly) strongly convex of order m at every $x^* \in X$, then the function h is said to be (strictly) strongly convex of order m on X.

Now, we present the following generalizations of generalized strong convexity.

Definition 2.2. *(Lin and Fukushima* [16]*). Let $m \geq 1$ be an integer number. A differentiable function $h : X \to \mathbb{R}$ is said to be (strictly) strongly pseudoconvex of order m at $x^* \in X$, if there exists a positive constant c such that*

$$(x - x^*)^T \nabla h(x^*) + c\|x - x^*\|^m \geq 0 \Rightarrow h(x)(>) \geq h(x^*), \forall\ x \in X,$$

or equivalently,

$$h(x)(\leq) < h(x^*) \Rightarrow (x - x^*)^T \nabla h(x^*) + c\|x - x^*\|^m < 0, \forall\ x \in X.$$

If the function h is (strictly) strongly pseudoconvex of order m at every $x^* \in X$, then the function h is said to be (strictly) strongly pseudoconvex of order m on X.

Definition 2.3. *(Lin and Fukushima* [16]*). Let $m \geq 1$ be an integer number. A differentiable function $h : X \to \mathbb{R}$ is said to be strongly quasiconvex of order m at $x^* \in X$, if there exists a positive constant c such that*

$$h(x) \leq h(x^*) \Rightarrow (x - x^*)^T \nabla h(x^*) + c\|x - x^*\|^m \leq 0, \forall\ x \in X,$$

or equivalently,

$$(x - x^*)^T \nabla h(x^*) + c\|x - x^*\|^m > 0 \Rightarrow h(x) > h(x^*), \forall\ x \in X.$$

If the function h is strongly quasiconvex of order m at every $x^* \in X$, then the function h is said to be strongly quasiconvex of order m on X.

Lemma 2.1. *(Generalized Schwartz Inequality). Let A be a positive semidefinite symmetric matrix of order n. Then, for all $x, v \in \mathbb{R}^n$,*

$$x^T Av \leq (x^T Ax)^{\frac{1}{2}} (v^T Av)^{\frac{1}{2}}.$$

The equality holds when $Ax = \xi Av$ for some $\xi \geq 0$. Evidently, if $(v^T Av)^{\frac{1}{2}} \leq 1$, we have

$$x^T Av \leq (x^T Ax)^{\frac{1}{2}}. \qquad (\mathbf{2}.1)$$

Following necessary conditions are the special case of Theorem 3.1 of Lai et al.[15], and is needed in the sequel:

Theorem 2.1. *(Necessary conditions). Let x^* be an optimal solution of the problem* (NP) *satisfying $x^{*T}Ax^* > 0$, and assume that $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent. Then there exist $(p, \lambda, \tilde{y}) \in M(x^*), v \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^k_+$ such that*

$$\sum_{i=1}^{p} \lambda_i \left\{\nabla f(x^*, \bar{y_i}) + Av\right\} + \nabla \sum_{j=1}^{k} \mu_j g_j(x^*) = 0, \qquad (\mathbf{2}.2)$$

$$\sum_{j=1}^{k} \mu_j g_j(x^*) = 0, \qquad (\mathbf{2}.3)$$

$$\lambda_i \geq 0,\ i = 1, 2, ..., p, \sum_{i=1}^{p} \lambda_i = 1, \qquad (\mathbf{2}.4)$$

$$v^T Av \leq 1,\ ({x^*}^T Ax^*)^{\frac{1}{2}} = {x^*}^T Av. \qquad (\mathbf{2}.5)$$

3. Sufficient conditions

In this section, we derive sufficient optimality conditions for minimax programming problem (NP) under proposed generalized convexity concept.

Theorem 3.1. *Let x^* be a feasible solution of the problem* (NP). *Assume that there exist $(p, \lambda, \tilde{y}) \in M(x^*), v \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^k_+$ satisfying relations (**2**.2)-(**2**.5). We define*

$$\Psi(.) = \sum_{i=1}^{p} \lambda_i \big(f(., \bar{y_i}) + (.)^T Av\big).$$

Furthermore, we assume that any one of the following conditions holds:

(*a*) $f(., \bar{y_i}) + (.)^T Av,\ i = 1, 2, ..., p$ *and* $\sum_{j=1}^{k} \mu_j g_j(.)$ *are strongly convex of order m at x^*;*

(*b*) $\Psi(.)$ *and* $\sum_{j=1}^{k} \mu_j g_j(.)$ *are strongly convex of order m at x^*;*

(*c*) $\Psi(.)$ *is strongly pseudoconvex of order m at x^* and* $\sum_{j=1}^{k} \mu_j g_j(.)$ *is strongly quasiconvex of order m at x^*;*

(*d*) $\Psi(.)$ *is strongly quasiconvex of order m at x^* and* $\sum_{j=1}^{k} \mu_j g_j(.)$ *is strictly strongly pseudoconvex of order m at x^*.*

Then x^* is an optimal solution of (NP).

Proof. Suppose, contrary to the result, that x^* is not an optimal solution of (NP). Then there exists $x \in \Omega$ such that

$$\sup_{y \in Y} \big(f(x, y) + (x^T Ax)^{\frac{1}{2}}\big) < \sup_{y \in Y} \big(f(x^*, y) + ({x^*}^T Ax^*)^{\frac{1}{2}}\big). \qquad (\mathbf{3}.1)$$

Since $\bar{y_i} \in Y(x^*)$ we have

$$\sup_{y \in Y} \big(f(x^*, y) + ({x^*}^T Ax^*)^{\frac{1}{2}}\big) = f(x^*, \bar{y_i}) + ({x^*}^T Ax^*)^{\frac{1}{2}}. \qquad (\mathbf{3}.2)$$

Also

$$f(x, \bar{y_i}) + (x^T Ax)^{\frac{1}{2}} \leq \sup_{y \in Y} \big(f(x, y) + (x^T Ax)^{\frac{1}{2}}\big). \qquad (\mathbf{3}.3)$$

Thus, from (**3**.1), (**3**.2), (**3**.3) we obtain

$$f(x, \bar{y_i}) + (x^T Ax)^{\frac{1}{2}} < f(x^*, \bar{y_i}) + ({x^*}^T Ax^*)^{\frac{1}{2}}. \qquad (\mathbf{3}.4)$$

From the inequalities (**2**.1), (**2**.4), (**2**.5) and (**3**.4) we get

$$\begin{aligned}\Psi(x) &= \sum_{i=1}^{p}\lambda_i\big(f(x,\bar{y_i})+x^TAv\big) \le \sum_{i=1}^{p}\lambda_i\big(f(x,\bar{y_i})+(x^TAx)^{\frac{1}{2}}\big)\\ &< \sum_{i=1}^{p}\lambda_i\big(f(x^*,\bar{y_i})+({x^*}^TAx^*)^{\frac{1}{2}}\big) = \sum_{i=1}^{p}\lambda_i\big(f(x^*,\bar{y_i})+{x^*}^TAv\big)\\ &= \Psi(x^*).\end{aligned}$$

Hence,

$$\Psi(x) < \Psi(x^*). \tag{3.5}$$

If condition (a) holds, then there exists a positive constant c such that

$$\begin{aligned}&f(x,\bar{y_i})+(x^TAv)-f(x^*,\bar{y_i})-({x^*}^TAv)\\ &\qquad \ge (x-x^*)^T\big(\nabla f(x^*,\bar{y_i})+Av\big)+c\|x-x^*\|^m,\ \forall\ i=1,2,...,p.\end{aligned}$$

Since $\sum\limits_{i=1}^{p}\lambda_i=1$, multiplying the above inequalities by λ_i and then summing, we obtain

$$\Psi(x)-\Psi(x^*) \ge (x-x^*)^T\sum_{i=1}^{p}\lambda_i\big(\nabla f(x^*,\bar{y_i})+Av\big)+c\|x-x^*\|^m,$$

which in turn, by using (**2**.2), implies

$$\Psi(x)-\Psi(x^*) \ge -(x-x^*)^T\sum_{j=1}^{k}\mu_j\nabla g_j(x^*)+c\|x-x^*\|^m. \tag{3.6}$$

On the other hand, by strong convexity of order m at x^* of $\sum\limits_{j=1}^{k}\mu_j g_j(.)$, there exists a positive constant $c^{'}$ such that

$$\sum_{j=1}^{k}\mu_j g_j(x)-\sum_{j=1}^{k}\mu_j g_j(x^*) \ge (x-x^*)^T\sum_{j=1}^{k}\mu_j\nabla g_j(x^*)+c^{'}\|x-x^*\|^m,$$

which by the feasibility of x in the problem (NP) and (**2**.3) yields

$$(x-x^*)^T\sum_{j=1}^{k}\mu_j\nabla g_j(x^*)+c^{'}\|x-x^*\|^m \le 0. \tag{3.7}$$

On adding inequalities (**3**.6) and (**3**.7) we obtain

$$\Psi(x)-\Psi(x^*) \ge (c^{'}+c)\|x-x^*\|^m.$$

Obviously $(c^{'}+c)>0$. Therefore, it follows from the above inequality

$$\Psi(x)-\Psi(x^*) \ge 0,$$

which contradicts inequality (**3**.5).

If condition (b) holds, then by strong convexity of order m at x^* of $\Psi(.)$, there exists a positive constant c such that

$$\Psi(x)-\Psi(x^*) \ge (x-x^*)^T\sum_{i=1}^{p}\lambda_i\big(\nabla f(x^*,\bar{y_i})+Av\big)+c\|x-x^*\|^m.$$

The above inequality along with (**2**.2) implies

$$\Psi(x) - \Psi(x^*) \geq -(x-x^*)^T \sum_{j=1}^{k} \mu_j \nabla g_j(x^*) + c\|x-x^*\|^m.$$

Now the proof follows on similar lines as in case (a).

If condition (c) holds, then by strong pseudoconvexity of order m at x^* of $\Psi(.)$ and inequality (**3**.5), it follows that there exists a positive constant c such that

$$(x-x^*)^T \nabla \Psi(x^*) + c\|x-x^*\|^m < 0,$$

which in turn along with equality (**2**.2) implies

$$(x-x^*)^T \sum_{j=1}^{k} \mu_j \nabla g_j(x^*) > c\|x-x^*\|^m \geq 0. \qquad (\mathbf{3}.8)$$

Now, from the feasibility of x in the problem (NP) and equality (**2**.3) we get

$$\sum_{j=1}^{k} \mu_j g_j(x) \leq \sum_{j=1}^{k} \mu_j g_j(x^*),$$

which by strong quasiconvexity of order m at x^* of $\sum_{j=1}^{k} \mu_j g_j(.)$ implies that there exists a positive constant $c^{'}$ such that

$$(x-x^*)^T \sum_{j=1}^{k} \mu_j g_j(x^*) + c^{'}\|x-x^*\|^m \leq 0.$$

Since $c^{'}\|x-x^*\|^m \geq 0$, the above inequality becomes

$$(x-x^*)^T \sum_{j=1}^{k} \mu_j \nabla g_j(x^*) \leq 0,$$

which contradicts inequality (**3**.8).

If condition (d) holds, then by strong quasiconvexity of order m at x^* of $\Psi(.)$ and inequality (**3**.5), it follows that there exists a positive constant $\tilde{c}$ such that

$$(x-x^*)^T \nabla \Psi(x^*) + \tilde{c}\|x-x^*\|^m < 0.$$

Rest of the proof follows on similar lines as in case of the condition (c). This completes the proof. $\square$

Now, we illustrate the Theorem **3**.1 by the following example.

Example 3.1. *Let* $X = \{x = (x_1, x_2) \in \mathbb{R}^2 : -1 < x_1 < 1 \text{ and } -1 < x_2 < 1\}$. *Consider the nondifferentiable minimax programming problem (NP), where* $Y = [-1, 1]$ *is a compact subset of* $\mathbb{R}$, $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}_{2\times 2}$ *is a positive semidefinite symmetric matrix. Let* $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuously differentiable functions defined by*

$$f(x,y) = \left(\frac{2x_1 + x_2}{x_2 + 1}\right)^2 y^2$$

and

$$g(x) = x_1^2 + x_1 x_2 + x_2^2$$

respectively. Obviously, the set of feasible solution of problem (NP) is $\Omega = \{x = (x_1, x_2) \in X : x_1^2 + x_1x_2 + x_2^2 \leq 0\}$*. By the definition of the set* $Y(x)$*, we see that* $Y(x) = \{y_1, y_2\} = \{-1, 1\}$ *and*

$$f(x, y_i) = \left(\frac{2x_1 + x_2}{x_2 + 1}\right)^2, i = 1, 2.$$

It is observed that the conditions (2.2)-(2.5) are satisfied for $(\lambda_1, \lambda_2, \mu) = (\frac{1}{3}, \frac{2}{3}, \frac{1}{2})$ *and* $v = (\frac{1}{2}, \frac{1}{2})$ *at* $x^* = (0, 0) \in \Omega$*. Also, it is easy to verify by means of Definition 2.1 that the functions* $f(x, y_i) + x^TAv$ *and* $\mu g(x)$ *are strongly convex of order* 2 *with* $(c, c') = (\frac{1}{4}, \frac{1}{4})$ *at* $x^* = (0, 0)$*. Therefore all the conditions of the Theorem 3.1 are satisfied. Thus* $x^* = (0, 0)$ *is an optimal solution for* (NP1).

4. First duality model

In this section we formulate the following Wolfe type dual (WD) for (NP) and derive duality results:

$$\text{(WD)} \quad \max_{(p,\lambda,\tilde{y})\in M(z)} \ \sup_{(z,v,\mu)\in F_1(p,\lambda,\tilde{y})} \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^TAv + \sum_{j=1}^{k} \mu_j g_j(z),$$

where $F_1(p, \lambda, \tilde{y})$ denotes the set of all $(z, v, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k_+$ satisfying

$$\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) = 0, \qquad (4.1)$$

$$v^TAv \leq 1. \qquad (4.2)$$

If for a triplet $(p, \lambda, \tilde{y}) \in M(z)$ the set $F_1(p, \lambda, \tilde{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Theorem 4.1. *(Weak duality). Let* x *and* $(z, v, \mu, p, \lambda, \tilde{y})$ *be feasible solutions of problems* (NP) *and* (WD)*, respectively. Assume that* $f(., \bar{y_i}) + (.)^TAv,\ i = 1, 2, ..., p$ *and* $g_j(.), j \in K$ *are strongly convex of order* m *at* z*. Then*

$$\sup_{y\in Y} \left(f(x, y) + (x^TAx)^{\frac{1}{2}}\right) \geq \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^TAv + \sum_{j=1}^{k} \mu_j g_j(z).$$

Proof. Suppose, contrary to the result, that

$$\sup_{y\in Y}(f(x, y) + (x^TAx)^{\frac{1}{2}}) < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^TAv + \sum_{j=1}^{k} \mu_j g_j(z).$$

Since $\bar{y_i} \in Y(x), i = 1, 2, ..., p$, the above inequality implies

$$f(x, \bar{y_i}) + (x^TAx)^{\frac{1}{2}} < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^TAv + \sum_{j=1}^{k} \mu_j g_j(z).$$

For $\lambda \in \mathbb{R}^p_+$ it follows that

$$\lambda_i \left\{ f(x, \bar{y_i}) + (x^TAx)^{\frac{1}{2}}) - \left(\sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^TAv + \sum_{j=1}^{k} \mu_j g_j(z) \right) \right\} \leq 0,$$

with at least one strict inequality, because $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \neq 0$. Taking summation over i and using $\sum\limits_{i=1}^{p} \lambda_i = 1$ we arrive at

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + (x^T Ax)^{\frac{1}{2}} < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j=1}^{k} \mu_j g_j(z).$$

The above inequality along with (**4**.2) and Lemma **2**.1 gives

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + x^T Av < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j=1}^{k} \mu_j g_j(z). \tag{4.3}$$

On the other hand, by strong convexity of order m at z of $f(., \bar{y_i}) + (.)^T Av,\ i = 1, 2, ..., p$ and $g_j(.), j \in K$, there exist positive constants c and $c_j^{'}, j \in K$, respectively, such that

$$f(x, \bar{y_i}) + x^T Av - f(z, \bar{y_i}) - z^T Av \geq (x - z)^T \big(\nabla f(z, \bar{y_i}) + Av\big) + c\|x - z\|^m \tag{4.4}$$

$$g_j(x) - g_j(z) \geq (x - z)^T \nabla g_j(z) + c_j^{'}\|x - z\|^m. \tag{4.5}$$

Since $\sum\limits_{i=1}^{p} \lambda_i = 1$, multiplying each inequality (**4**.4) by $\lambda_i, i = 1, 2, ..., p$, and each inequality (**4**.5) by $\mu_j, j = 1, 2, ..., k$, and then taking sum we get

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + x^T Av + \sum_{j=1}^{k} \mu_j g_j(x) - \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) - z^T Av - \sum_{j=1}^{k} \mu_j g_j(z)$$

$$\geq (x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) \right) + \left(c + \sum_{j=1}^{k} \mu_j c_j^{'} \right) \|x - z\|^m. \tag{4.6}$$

From (**4**.3), (**4**.6) and the feasibility of x in the problem (NP) we get

$$(x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) \right) + \left(c + \sum_{j=1}^{k} \mu_j c_j^{'} \right) \|x - z\|^m < 0.$$

Obviously $\left(c + \sum\limits_{j=1}^{k} \mu_j c_j^{'} \right) > 0$. Therefore, it follows from the above inequality

$$(x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) \right) < 0,$$

which contradicts inequality (**4**.1). This completes the proof. □

Theorem 4.2. *(Weak duality). Let x and $(z, v, \mu, p, \lambda, \tilde{y})$ be feasible solutions of problems* (NP) *and* (WD)*, respectively. Assume that $\sum\limits_{i=1}^{p} \lambda_i f(., \bar{y_i}) + (.)^T Av + \sum\limits_{j=1}^{k} \mu_j g_j(.)$ are strongly pseudoconvex of order m at z. Then*

$$\sup_{y \in Y} (f(x, y) + (x^T Ax)^{\frac{1}{2}}) \geq \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j=1}^{k} \mu_j g_j(z).$$

Proof. Suppose, contrary to the result, that

$$\sup_{y \in Y} (f(x, y) + (x^T Ax)^{\frac{1}{2}}) < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j=1}^{k} \mu_j g_j(z).$$

Now, proceeding on the same lines as in the Theorem **4**.1, we see that

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + x^T Av < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j=1}^{k} \mu_j g_j(z). \qquad (\mathbf{4}.7)$$

Using the feasibility of x in problem (NP) and $\mu_j \geq 0,\ j = 1, 2, ..., k$ we have

$$\sum_{j=1}^{k} \mu_j g_j(x) \leq 0. \qquad (\mathbf{4}.8)$$

By (**4**.7) and (**4**.8) it follows that

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + x^T Av + \sum_{j=1}^{k} \mu_j g_j(x) < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j=1}^{k} \mu_j g_j(z),$$

which by strong pseudoconvexity of order m at z of $\sum\limits_{i=1}^{p} \lambda_i f(., \bar{y_i}) + (.)^T Av + \sum\limits_{j=1}^{k} \mu_j g(.)$ implies that there exists a positive constant c such that

$$(x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) \right) + c\|x - z\|^m < 0.$$

Obviously $c > 0$. Therefore, it follows from the above inequality

$$(x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) \right) < 0,$$

which contradicts inequality (**4**.1). This completes the proof. □

Theorem 4.3. *(Strong duality). Let x^* be an optimal solution for* (NP) *and the vectors $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent. Then, there exist $(p^*, \lambda^*, \tilde{y^*}) \in M(x^*)$ and $(x^*, v^*, \mu^*) \in F_1(p^*, \lambda^*, \tilde{y^*})$ such that $(x^*, v^*, \mu^*, p^*, \lambda^*, \tilde{y^*})$ is a feasible solution of* (WD)*. In addition, if the assumptions of weak duality theorem (Theorem* **4**.1 *or Theorem* **4**.2*) holds for all feasible solutions of* (WD)*, then the point $(x^*, v^*, \mu^*, p^*, \lambda^*, \tilde{y^*})$ is an optimal solution of* (WD) *and the two problems* (NP) *and* (WD) *have the same optimal values.*

Proof. Since x^* is an optimal solution of problem (NP) and the vectors $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent, then by Theorem **2**.1, there exist $(p^*, \lambda^*, \tilde{y^*}) \in M(x^*)$ and $(x^*, v^*, \mu^*) \in F_1(p^*, \lambda^*, \tilde{y^*})$ such that $(x^*, v^*, \mu^*, p^*, \lambda^*, \tilde{y^*})$ is a feasible solution of (WD) and the corresponding objective of values of (NP) and (WD) are equal. The optimality of this feasible solution for (WD) thus follows from weak duality Theorem **4**.1 or Theorem **4**.2. □

Theorem 4.4. *(Strict converse duality). Let x^* and $(z^*, v^*, \mu^*, p^*, \lambda^*, \tilde{y^*})$ be feasible solutions of* (NP) *and* (WD)*, respectively. Assume that $f(., \bar{y_i^*}) + (.)^T Av,\ i = 1, 2, ..., p$ and $g_j(.), j \in K$ are strictly strongly convex of order m at z^*. Further, assume that $\nabla g(x^*), j \in J(x^*)$ are linear independent. Then, $z^* = x^*$; that is, z^* is an optimal solution of* (NP)*.*

Proof. Suppose, contrary to the result, that $z^* \neq x^*$. From strong duality Theorem **4**.3, we reach

$$\sup_{y \in Y} \left(f(x^*, \bar{y^*}) + ({x^*}^T Ax^*)^{\frac{1}{2}} \right) = \sum_{i=1}^{p^*} \lambda_i^* f(z^*, \bar{y_i^*}) + {z^*}^T Av^* + \sum_{j=1}^{k} \mu_j^* g_j(z^*).$$

Thus, we have

$$f(x^*,\bar{y_i^*})+(x^{*^T}Ax^*)^{\frac{1}{2}}\leq\sum_{i=1}^{p^*}\lambda_i^*f(z^*,\bar{y_i^*})+z^{*^T}Av^*+\sum_{j=1}^{k}\mu_j^*g_j(z^*),$$

for all $\bar{y_i^*}\in Y(x^*),\ i=1,2,...,p^*$.
Now, proceeding on the same lines as in the Theorem **4**.1, using strictly strongly convex of order m we get

$$(x^*-z^*)^T\left(\sum_{i=1}^{p^*}\lambda_i^*\nabla f(z^*,\bar{y_i^*})+Av^*+\sum_{j=1}^{k}\mu_j^*\nabla g_j(z^*)\right)<0,$$

which contradicts inequality (**4**.1). This completes the proof. □

Theorem 4.5. *(Strict converse duality). Let x^* and $(z^*,v^*,\mu^*,p^*,\lambda^*,\tilde{y^*})$ be feasible solutions of* (NP) *and* (WD)*, respectively. Assume that* $\sum_{i=1}^{p^*}\lambda_i^*f(.,\bar{y_i^*})+(.)^TAv^*+\sum_{j=1}^{k}\mu_j^*g_j(.)$ *are strictly strongly pseudoconvex of order m at z^*. Further, assume that $\nabla g(x^*), j\in J(x^*)$ are linear independent. Then, $z^*=x^*$; that is, z^* is an optimal solution of* (NP).

Proof. The proof is similar to that of the above theorem and hence being omitted. □

5. Second duality model

In this section, we formulate the following general Mond-Weir type dual (MD) for (NP) and derive duality results:

$$\text{(MD)}\quad\max_{(p,\lambda,\tilde{y})\in M(z)}\ \sup_{(z,v,\mu)\in F_2(p,\lambda,\tilde{y})}\ \sum_{i=1}^{p}\lambda_if(z,\bar{y}_i)+z^TAv+\sum_{j\in J_0}\mu_jg_j(z)$$

where $F_2(p,\lambda,\tilde{y})$ denotes the set of all $(z,v,\mu)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+^k$ satisfying

$$\sum_{i=1}^{p}\lambda_i\nabla f(z,\bar{y}_i)+Av+\sum_{j=1}^{k}\mu_j\nabla g_j(z)=0, \tag{5.1}$$

$$\sum_{j\in J_\gamma}\mu_jg_j(z)\geq0,\gamma=1,2,...,t, \tag{5.2}$$

$$v^TAv\leq1,\ (z^TAz)^{\frac{1}{2}}=z^TAv, \tag{5.3}$$

where $J_\gamma\subseteq K=\{1,2,...,k\},\gamma=0,1,2,...,t$ with $\cup_{\gamma=0}^{t}J_\beta=K$ and $J_\beta\cap J_\gamma\neq\phi$, if $\beta\neq\gamma$. If for a triplet $(p,\lambda,\tilde{y})\in M(z)$, the set $F_2(p,\lambda,\tilde{y})=\emptyset$, then we define the supremum over it to be $-\infty$.

Remark 5.1. *If $J_0=K,\ J_\gamma=\phi\ (1\leq\gamma\leq\beta)$, then (MD) reduces to (WD).*

Theorem 5.1. *(Weak duality). Let x and $(z,v,\mu,p,\lambda,\tilde{y})$ be feasible solutions of* (NP) *and* (MD)*, respectively. Assume that* $\sum_{i=1}^{p}\lambda_if(.,\bar{y}_i)+(.)^TAv+\sum_{j\in J_0}\mu_jg_j(.)$ *and* $\sum_{j\in J_\gamma}\mu_jg_j(.)$ *are strongly convex of order m at z. Then*

$$\sup_{y\in Y}\left(f(x,y)+(x^TAx)^{\frac{1}{2}}\right)\geq\sum_{i=1}^{p}\lambda_if(z,\bar{y}_i)+z^TAv+\sum_{j\in J_0}\mu_jg_j(z).$$

Proof. Suppose, contrary to the result, that

$$\sup_{y\in Y}\left(f(x,y)+(x^TAx)^{\frac{1}{2}}\right)<\sum_{i=1}^{p}\lambda_if(z,\bar{y}_i)+z^TAv+\sum_{j\in J_0}\mu_jg_j(z).$$

Since $\bar{y_i} \in Y(x),\ i = 1, 2, ..., p$, the above inequality implies

$$f(x, \bar{y_i}) + (x^T Ax)^{\frac{1}{2}} < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j \in J_0} \mu_j g_j(z).$$

For $\lambda \in \mathbb{R}^p_+$, from above inequality we arrive at

$$\lambda_i \left[f(x, \bar{y_i}) + (x^T Ax)^{\frac{1}{2}} - \left(\sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j \in J_0} \mu_j g_j(z) \right) \right] \leq 0,$$

with at least one strict inequality, because $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \neq 0$. Taking summation over i and using $\sum_{i=1}^{p} \lambda_i = 1$ we have

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + (x^T Ax)^{\frac{1}{2}} < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j \in J_0} \mu_j g_j(z).$$

The above inequality along with (**5**.3) and Lemma **2**.1 implies

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + x^T Av < \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) + z^T Av + \sum_{j \in J_0} \mu_j g_j(z). \tag{5.4}$$

Also, from (**5**.2) we have

$$- \sum_{j \in J_\gamma} \mu_j g_j(z) \leq 0. \tag{5.5}$$

On the other hand, by strong convexity of order m at z of $\sum_{i=1}^{p} \lambda_i f(., \bar{y_i}) + (.)^T Av + \sum_{j \in J_0} \mu_j g_j(.)$ and $\sum_{j \in J_\gamma} \mu_j g_j(.)$, there exist positive constants c and $c^{'}$ respectively such that

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + x^T Av + \sum_{j \in J_0} \mu_j g_j(x) - \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) - z^T Av - \sum_{j \in J_0} \mu_j g_j(z)$$
$$\geq (x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j \in J_0} \mu_j \nabla g_j(z) \right) + c\|x - z\|^m,$$

$$\sum_{j \in J_\gamma} \mu_j g_j(x) - \sum_{j \in J_\gamma} \mu_j g_j(z) \geq (x - z)^T \left(\sum_{j \in J_\gamma} \mu_j \nabla g_j(z) \right) + c^{'} \|x - z\|^m.$$

On adding the above two inequalities we obtain

$$\sum_{i=1}^{p} \lambda_i f(x, \bar{y_i}) + x^T Av + \sum_{j=1}^{k} \mu_j g_j(x) - \sum_{i=1}^{p} \lambda_i f(z, \bar{y_i}) - z^T Av - \sum_{j \in J_0} \mu_j g_j(z) - \sum_{j \in J_\gamma} \mu_j g_j(z)$$
$$\geq (x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) \right) + (c + c^{'})\|x - z\|^m. \tag{5.6}$$

From (**5**.4), (**5**.5), (**5**.6) and the feasibility of x in the problem (NP) we get

$$(x - z)^T \left(\sum_{i=1}^{p} \lambda_i \nabla f(z, \bar{y_i}) + Av + \sum_{j=1}^{k} \mu_j \nabla g_j(z) \right) + (c + c^{'})\|x - z\|^m < 0.$$

Obviously $(c+c^{'})>0$. Therefore, it follows from the above inequality

$$(x-z)^T\left(\sum_{i=1}^{p}\lambda_i\nabla f(z,\bar{y_i})+Av+\sum_{j=1}^{k}\mu_j\nabla g_j(z)\right)<0,$$

which contradicts inequality (**5**.1). This completes the proof. □

Theorem 5.2. *(Weak duality). Let* x *and* $(z,v,\mu,p,\lambda,\tilde{y})$ *be feasible solution of* (NP) *and* (MD)*, respectively. Assume that* $\sum_{i=1}^{p}\lambda_i f(.,\bar{y_i})+(.)^T Av+\sum_{j=1}^{k}\mu_j g(.)$ *is strongly pseudoconvex of order* m *at* z*. Then*

$$\sup_{y\in Y}(f(x,y)+(x^TAx)^{\frac{1}{2}})\geq\sum_{i=1}^{p}\lambda_i f(z,\bar{y_i})+z^TAv+\sum_{j\in J_0}\mu_j g_j(z).$$

Proof. Suppose, contrary to the result, that

$$\sup_{y\in Y}(f(x,y)+(x^TAx)^{\frac{1}{2}})<\sum_{i=1}^{p}\lambda_i f(z,\bar{y_i})+z^TAv+\sum_{j=1}^{k}\mu_j g_j(z).$$

Now, proceeding on same line as in the Theorem **5**.1, we see that

$$\sum_{i=1}^{p}\lambda_i f(x,\bar{y_i})+x^TAv<\sum_{i=1}^{p}\lambda_i f(z,\bar{y_i})+z^TAv+\sum_{j\in J_0}\mu_j g_j(z). \quad (\mathbf{5}.7)$$

Using the feasibility of x in the problem (NP) and $\mu_j,\ j=1,2,...,k$, we get

$$\sum_{j=1}^{k}\mu_j g_j(x)\leq 0. \quad (\mathbf{5}.8)$$

By (**5**.2), (**5**.7) and (**5**.8) it follows that

$$\sum_{i=1}^{p}\lambda_i f(x,\bar{y_i})+x^TAv+\sum_{j=1}^{k}\mu_j g_j(x)<\sum_{i=1}^{p}\lambda_i f(z,\bar{y_i})+z^TAv+\sum_{j=1}^{k}\mu_j g_j(z),$$

which by strong pseudoconvexity of order m at z of $\sum_{i=1}^{p}\lambda_i f(.,\bar{y_i})+(.)^TAv+\sum_{j=1}^{k}\mu_j g_j(.)$ implies that there exists a positive constant c such that

$$(x-z)^T\left(\sum_{i=1}^{p}\lambda_i\nabla f(z,\bar{y_i})+Av+\sum_{j=1}^{k}\mu_j\nabla g_j(z)\right)+c\|x-z\|^m<0.$$

Obviously $c>0$. Therefore, it follows from the above inequality

$$(x-z)^T\left(\sum_{i=1}^{p}\lambda_i\nabla f(z,\bar{y_i})+Av+\sum_{j=1}^{k}\mu_j\nabla g_j(z)\right)<0,$$

which contradicts inequality (**5**.1). This completes the proof. □

Theorem 5.3. *(Strong duality). Let* x^* *be an optimal solution for* (NP) *and the vectors* $\nabla g_j(x^*), j\in J(x^*)$ *are linearly independent. Then there exist* $(p^*,\lambda^*,\tilde{y^*})\in M(x^*)$ *and* $(x^*,v^*,\mu^*)\in F_2(p^*,\lambda^*,\tilde{y^*})$ *such that* $(x^*,v^*,\mu^*,p^*,\lambda^*,\tilde{y^*})$ *is a feasible solution of* (MD)*. In addition, if the assumptions of weak duality theorem (Theorem* **5**.*1 or Theorem* **5**.*2) hold for all feasible solutions of* (MD)*, then the point* $(x^*,v^*,\mu^*,p^*,\lambda^*,\tilde{y^*})$ *is an optimal solution of* (MD) *and the two problems* (NP) *and* (MD) *have the same optimal value.*

Proof. The proof follows on similar lines of Theorems **4**.3. □

Theorem 5.4. *(Strict converse duality). Let x^* and $(z^*, v^*, \mu^*, p^*, \lambda^*, \tilde{y^*})$ be feasible solutions of* (NP) *and* (MD)*, respectively. Assume that $\sum_{i=1}^{p^*} \lambda_i f(., \bar{y_i^*}) + (.)^T Av + \sum_{j\in J_0} \mu_j g_j(.)$ and $\sum_{j\in J_\gamma} \mu_j g_j(.)$ are strictly strongly convex of order m at z. Further, assume that $\nabla g(x^*)$, $j \in J(x^*)$ are linear independent. Then, $z^* = x^*$; that is, z^* is an optimality solution of* (NP).

Proof. The proof follows on similar lines of Theorems **4**.4. □

Remark 5.2. *Let $A = 0$. Then (NP) and (WD) become the problems proposed by Tanimoto* [20].

6. Conclusion

In the present work, we have established the sufficient optimality conditions for a nondifferentiable minimax programming problem under strong convexity/generalized convexity of order m. Weak, strong and strict converse duality results for two types of duals viz., Wolfe and general Mond-Weir type dual problems were also established by using the aforesaid convexity/generalized convexity concepts. It seems that the techniques developed in this paper can also be used to provide similarly results for a nonsmooth minimax programming problem. This will be the future task of the authors.

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