COMPUTATION OF HURST EXPONENT OF TIME SERIES USING DELAYED (LOG-) RETURNS. APPLICATION TO ESTIMATING THE FINANCIAL VOLATILITY

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We present a simple and straightforward method for computing Hurst exponent of fractal, self-similar time-series by direct use of the defining relationship. It is based on a set of series derived from the original time series constructed considering log-returns for increasing delays. The method can be applied to self-similar time series of any kind, irrespective of their origin. In the present study we only consider financial time series and the computed Hurst exponent is used for estimation of market volatility. As case study, the exchange rates of ROL, CZK, GBP and SEK versus US Dollar and Euro are considered.

Keywords: Hurst exponent, fractal analysis, financial time series, volatility

1. Introduction

The Hurst exponent, introduced by H. E. Hurst [1] was later proposed for use in fractal analysis [2,3], and used to many research fields such as biology [4,5], geophysical dynamics [6,8], turbulence in fluids and plasmas [9,10]. Recently, the fractal analysis has become popular in the finance research, particularly in the context of Econophysics [11], a relatively new area of study, developed by cooperation between economists, mathematicians and physicists. It applies ideas, methods and models of statistical physics and complexity theory to analyze data from economical phenomena. Especially it extended the fractal analysis to economic-financial dynamics [12-16].

The Hurst exponent provides a measure for long term memory of time series, very useful in forecasting, where the first question we want to answer is whether the time series under study is more or less predictable.

In this work we present a new simple and straightforward method for computing Hurst exponent of fractal, self-similar time-series by direct use of the defining relationship. It is based on a set of series derived from the original time series constructed considering returns or log-returns for increasing delays. The

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method can be applied to self-similar time series of any kind, irrespective of their origin. In the present study we only consider financial time series and the computed Hurst exponent is used for estimation of market volatility.

Every economic agent has a certain strategy to obtain profit. The greater the company, the longer the time horizon of the plan, and the higher the impact of the decisions taken at the top management level. Clearly, the government policy, central bank policy and powerful corporations represent a biasing presence in all planning actions and decisions of the business management of the firm in all its components such as financial, labor or risk management. Risks can come from uncertainty in financial markets, project failures, legal liabilities, credit risk, accidents, natural situations or disasters, as well as deliberate attack from an adversary, or events of uncertain origin or unpredictable cause [17].

An important role in management is occupied by the time series analysis as any industrial or agricultural processes, banking and financial activity, insurance and pension systems or markets and stock exchange can be characterized by a succession of data such as daily output, foreign exchange rates, monthly sales volumes, etc. The analysis is based on the hypothesis that a realistic, efficient business management has to rely on previous data, usually organized as time series. These show the past evolution of the own business as well as those of partner or competitive businesses. The degrees of correlation between them can be an important factor in the decision making process.

Analysis of economic and financial time series for identifying their characteristic parameters, particularly the volatility is of great importance in the risk management and risk prevention activity. Volatility is the up-and-down movement of the market. Any movement up or down from its expectation, measured by the mean of the time series of that particular stock or exchange rate, is the volatility [18].

The analysis in the present work is limited to financial time series of the exchange rates for four currencies: Romanian Leu (ROL), Czech Koruna (CZK), British Pound (GBP) and Swedish Krona (SEK) versus US Dollar (USD) and versus Euro (EUR). However, this type of analysis was already used to exchanges rates particularly involving ROL [19], and is also applicable to other types of financial series such as market daily sales volume, stock market closing index, monthly production output of a factory, etc.

The studied daily exchange rates were taken from the site FOREX Trading and Exchange Rates Services (OANDA) and correspond to the interval June 1999-March 2013, i.e. 5083 values [20].

2. Theoretical considerations

A geometrical structure is called self-similar if it is exactly or approximately similar to its parts. Many objects of the real world are statistically self-similar *i.e.* their parts have the same statistical properties as the whole structure. Self-similarity is a characteristic property of fractal objects [21,22].

The self-similarity property is also a characteristic of stochastic processses. Stochastic self-similarity is defined as follows: a real-valued process $\{X(t), t \in R\}$ is statistically self-similar (or self-afine) of index $H > 0$, if for any $a > 0$, the statistical distribution of $\{X(at), t \in R\}$ is identical to the distribution of $\{a^H X(t), t \in R\}$. To put it slightly differently, if $t_1, t_2, \cdots, t_n \in R$ then

$$\left(a^H X(t_1), a^H X(t_2), \cdots, a^H X(t_n)\right) \overset{d}{=} \left(X(at_1), X(at_2), \cdots, X(at_n)\right) \tag{1}$$

i.e. the distributions of values of the two processes are identical. For given $t \in R$,

$$a^H X(t) \overset{d}{=} X(at). \tag{2}$$

The symbol $\overset{d}{=}$ should be read ”identity in distribution”, and signifies statistical identity, i.e. that the functions in the two sides of the equation are statistically indiscernable. The impossibility to verify this identity for all the moments of the two distributions leads to a more loose criteria which requests that the identity must be satisfied only by the first two moments: the mean and the variance.

If the scales on the two „directions” are different, k_x for the parameter direction, and k_t for the time direction, then $k_x = a^H$ and $k_t = a$ such that, the ratio

$$H = \frac{\log k_x}{\log k_t} \tag{3}$$

is defined for the respective stochastic process. The index H is called Hurst exponent, because its introduction was suggested by Hurst’s study on the dynamics of river Nile flow [1].

Unlike the self-similar curves (such as *e.g.* the von Koch curve [9]), where the two directions are both spatial variables, characterized by the same value of the scaling factor, a function associated to a stochastic process is characterized by different scaling factors for the two (different) variables, c for t and c^H for X. Usually, the structures with nonuniform scaling are called *self-afine*. However, very frequently, this distinction is not observed, and the name self-similarity is used for both situations.

Perhaps the earliest example of a self-similar series is the one-dimensional Brownian motion. It represents the successive positions of a small particle immersed in a fluid projected on a particular direction, say the x axis, generating a series of data $x_i = x(t_i)$ $\left(i = 1,2,\cdots,N\right)$. The theoretical treatment by Langevin and Einstein at the beginning of the 20-th Century [23], demonstrated that the variance of this series depends on the time interval between successive observations $\Delta t = t_{i+1} - t_i$ in the form

$$\sigma^2 = \left\langle \left(\Delta x\right)^2 \right\rangle \sim \Delta t\,, \tag{4}$$

where the angular parantheses describe an ensamble average.

In many fields of science, as well as in the case of algorithmic modeling, for a prescribed stochastic process, an arbitrary number of realizations can be generated as particular time series, forming a statistical ensemble. In the case of real economic-financial processes this is impossible, and as consequence, each time series has to be considered as representing a unique realization of the underlying stochastic process and the ensemble average in Eq.(4) has to be removed. Taking into consideration the alteration of precision on Hurst exponent computation with shortening of the series, irrespective of the used algorithm, we do not consider breaking the series in partial segments as recommended in [24].

According to (3), the Hurst exponent for the Brownian process is

$$H = \frac{1}{2}\,. \tag{5}$$

The succesive variations Δx are totally un-correlated and constitute a Gaussian white noise characterized by zero mean and normal distribution.

The special interest of the Brownian process for economic analysis and finance originates in the fact that, at about the same time with Langevin and Einstein, Bachelier [25], observed that the output of industrial and financial processes registered at constant time intervals generates a data sequence very similar to a Brownian motion time series.

Later, at mid 20-th Century, Mandelbrot [2,3,21] defined the so called “fractional Brownian motion” characterized by Hurst exponents with values in the interval (0,1), extensively used as models for financial processes.

For a fractional Brownian motion of index $0 < H < 1$, Eq. (4) becomes

$$\sigma \sim \left(\Delta t\right)^H \tag{6}$$

with H=1/2 for regular Brownian motion.

The Hurst exponent refers to aspects related to the memory of the process. Thus, if $0.5 < H < 1$, the system is characterized by long-memory effects demonstrating *persistence* (an incresing is more likely to be followed by another incresing than by a decreasing) while systems with $0 < H < 0.5$ are *antipersistent* (an incresing is more likely to be followed by a decreasing than by another increasing). The Hurst exponent can be conceived as measure for the degree of irregularity of the graph of a time series: a large value of H corresponds to a time series with a graph smoother than another one characterized by lower value of H.

3. New method to estimate the Hurst exponent

In this analysis we introduce a method for estimation of the Hurst exponent for any particular time series, based on Eq. (6). The same equation was used in [24] but in a different type of analysis. We consider the following procedure. Let’s consider a financial time series expressed as $x_1, x_{2,} \cdots, x_N$. Initially, the series of variation for a delay of one time step is computed according to the relationship

$$y_n^{(1)} = x_{n+1} - x_n \quad (n = 1,2,\cdots N-1) \tag{7}$$

This is generalized to a delay of an arbitrary number k of time steps

$$y_n^{(k)} = x_{n+k} - x_n \quad (n = 1,2,\cdots N-k). \tag{8}$$

In the following, instead of “variation for a delay of k time steps” for short, the name “(delayed) return”, will be adopted. Then, the series of log-returns which, in financial analysis are usually called “returns” is generated according to

$$z_n^{(1)} = \log x_{n+1} - \log x_n \quad (n = 1,2,\cdots N-1) \tag{9}$$

and generalized for an arbitrary k step delay

$$z_n^{(k)} = \log x_{n+k} - \log x_n \quad (n = 1,2,\cdots N-k). \tag{10}$$

Subsequently, the standard deviation for the obtained series is :

$$\sigma_y^{(k)} = \sqrt{\frac{1}{N-k}\sum_n \left(y_n^{(k)} - \overline{y^{(k)}}\right)^2} \qquad \sigma_z^{(k)} = \sqrt{\frac{1}{N-k}\sum_n \left(z_n^{(k)} - \overline{z^{(k)}}\right)^2} \tag{11}$$

where the average values (the means) are given by

$$\overline{y^{(k)}} = \frac{1}{N-k}\sum_n y_n^{(k)}, \quad \text{and} \quad \overline{z^{(k)}} = \frac{1}{N-k}\sum_n z_n^{(k)} \quad (n = 1,2,\cdots, N-k) \tag{12}$$

respectively.

With increasing the k, the (log-)return series show larger and larger fluctuations or equivalently fatter and fatter frequency distributions (histograms). If in the absence of averaging requested by Eq. (4) the dependence of this increasing on the delay is power law type, then, the graph $\log \sigma_y^{(k)}$ versus $\log k$ will be a straight line and the value of the Hurst exponent is computed as its slope.

4. Results and discussions

In order to apply Eq.(6) we make a graph of $\log \sigma_y^{(k)}$ versus $\log k$ for all considered values of k, including k=1. The graph corresponding to the ROL/EUR exchange rate time series is presented in figure 1.

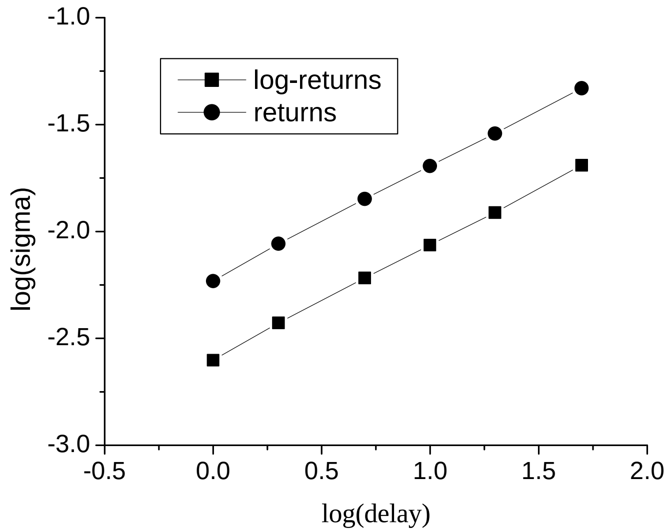

Fig.1. Graph for computation of Hurst exponent as slopes of the log σ versus log k curves: H(returns)=0.568; H(log-returns)=0.565

As for a given series, $\sigma_y^{(k)}/\sigma_z^{(k)}$ is a number, it is to be expected that the same type of dependence (6) should be satisfied when the standard deviation of the return series is replaced by the standard deviation of the log-returns one, certainly with a different proportionality constant. Intuitively, it is clear that the results obtained using the two types of return should be consistent, because the distribution of the data in the two wings of the frequency distribution is the same and this is related to the persistence in the series [26]. Figure 1 illustrates this situation. At the same time it shows that the graph is with very good approximation a straight line with equal slope, in support of our hypothesis. This demonstrates that by the proposed method, the Hurst exponent can be computed using either the returns series or the log-returns one.

Next, we applied the method for the exchange rate time series of ROL, CZK, GBP and SEK versus EUR and Versus USD, using the associated log-returns series. The results are graphically presented in Fig.2 for the following values of the delay: k=1, 2, 5, 10, 20, and 50. We observe that in all cases the graph of $\log\sigma_y^{(k)}$ versus $\log k$ is extremely well approximated as a linear dependence. According to the theoretical aspects previously presented, the slope of the line has to be interpreted as the value of the Hurst exponent of the x_n series. The numerical results are synthesized in Table 1.

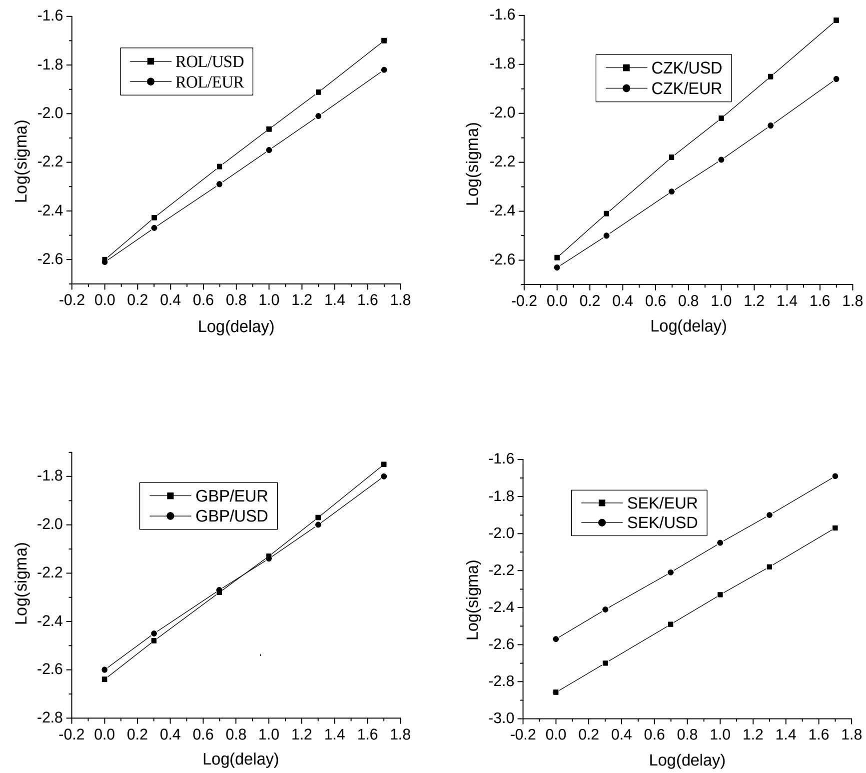

Fig.1. Graphs for computation of Hurst exponent as slopes of the log σ versus log k curves for eight exchange rate time series as specified on each graph

We can observe two different situations: in the case of the two former socialist countries, the Hurst exponent for the exchange rate against Euro is smaller than the Hurst exponent for the exchange rate versus USD while, for the two analyzed western countries the Hurst exponent has the same value, within computational errors, irrespective of the currency versus which the exchange rate is considered. This can be interpreted as consequence of the attitude of the respective economic-financial community towards the two reference currencies.

5. Volatility and computation of volatility for specified time horizon

A measure of risk is based on the volatility (standard deviation) of the asset log-return series. In finance, volatility is a statistical measure for the tendency of a security's price to change over time. For technical reasons, volatility is usually computed based on log-return rather than return series computed for a one time step delay. For example, volatility appears in option pricing formulas, where it quantifies the variability of the underlying asset return from now to the expiration of the option.

In most applications based on time series analysis, volatility is estimated for particular time horizons such as a day, a month or a year. The historic volatility for a time series representing daily data is denoted σ^1 and is also known as daily volatility. If we are interested in the volatility for a time horizon of *T* days then, according to equation (6) this is computed as

$$\sigma_T = \sigma^1 T^H \tag{13}$$

where, for simplicity's sake, $H = 1/2$ is frequently considered.

For a time horizon of one year, the annualized volatility is given by Eq. (13) with $T = N_d$ - the average number of trading days in a year. It is well established that many financial time series are better approximated as fractional Brownian motion of a certain index *H*. In this case, the power in the time horizon volatility (13) must be computed using the corresponding *H*. The right-most column of Table 1 presents the annualized volatility computed considering that the average number of trading days in a year is $N_d = 252$.

Table 1– *Computational results (maximum delay k=50)*

Series	Slope	Intercept	σ_1	Annualized volatility
ROL/EUR	0.468	-2.60	0.00251	0.0334
ROL/USD	0.565	-2.61	0.00245	0.0557
CZK/EUR	0.452	-2.63	0.00234	0.0284
CZK/USD	0.564	-2.59	0.00257	0.0581
GBP/EUR	0.532	-2.64	0.00229	0.0433
GBP/USD	0.528	-2.61	0.00245	0.0454
SEK/EUR	0.523	-2.85	0.00141	0.0254
SEK/USD	0.524	-2.59	0.00257	0.0466

In order to find out the range of acceptable delays we computed two additional points, for delays of 100 and 150. While the point corresponding to the *k*=100 is well situated along the line defined by the previous points, the point for the delay 150 is slightly but visibly off this line. It is to be concluded that for very good results, the maximal value of the delay should not exceed 2% of the length of the analyzed series.

For estimation of the value of the computed Hurst exponents for prediction, a number of 300 Brownian series with the same length of slightly over 5000 data were generated, and their Hurst exponent computed using the proposed method. We found a distribution of frequencies close to Gaussian with average

$H_{ave} = 0.496$ and standard deviation $\sigma_H = 0.032$. It means that Hurst exponents of the analyzed series in the interval between 0.464 and 0.528, with probability of 68.3% coincide with Hurst exponent for totally uncorrelated series. As seen from Table 1, the Hurst exponent of some of the analyzed series are outside this interval, indicating the presence of a limited degree of correlation.

6. Conclusion

Computing of Hurst exponent of a time series gives valuable information on the predictability in the process that generated it.

The new method presented in this work is based on the analysis of the dependence of log-returns on the delay. The Hurst exponent for each time series is computed as the slope of the linear fit of the log-log graph of the standard deviation (volatility) of the log-returns series versus the time delay. We think that the very good approximation as a straight line of this graph is due to some compensation for the absence of the averaging in Eq.(6) given by computation of returns for the entire original series. The method can be applied to any type of time series however, for time series that also contain negative data only the analysis based on returns is accessible.

Application to financial time series shows that the behavior for each currency rate can be correlated with the characteristics of the corresponding economical system. The foreign exchange time series for the two former socialist countries are different from the Western countries. The ROL and CZK exchange rates are slightly predictable while GBP and SWK exchange rates present no predictability. Also while the rate of former socialist countries against USD shows some persistence, their rate against EUR are slightly anti-persistent.

Application of the results of the analysis to the study of risk management, particularly to the FOREX trade is of interest. In this context volatility refers to the amount of uncertainty or risk involved with the size of changes in a currency exchange rate. As high volatility means that the price of the currency can change dramatically over a short time period in either direction, volatility is often used to quantify the risk of the currency pair over the time period of interest. The higher the volatility, the riskier the trading of the currency pair is.

Volatility is usually considered a negative as it represents uncertainty and risk. However, higher volatility can make FOREX trading more attractive to the market players. High volatility markets are especially interesting to traders looking for immediate profit, while for long term, buy and hold investors, it is a discouraging factor. In the FOREX trading, usually shorter time horizons are of interest, such as a few days, a week or a month.

We found good consistency of the proposed method for Hurst exponent computation with the detrended fluctuation analysis method. However, its the consistency with other methods and the applicability to other types of time series will be presented in another paper.

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