A SHORT NOTE ON APPROXIMATIONS IN A RING BY USING A NEIGHBORHOOD SYSTEM AS A GENERALIZATION OF PAWLAK'S APPROXIMATIONS

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Let R be a ring. We consider the relation α and its transitive closure α^*. The relation α is the smallest equivalence relation on R so that R/α^* is a commutative ring. Based on the relation α, we define a neighborhood system for each element of R, and we present a general framework of the study of approximations in rings. The connections between rings and operators are examined.

1. Introduction

We are familiar with the binary operations of addition and multiplication among many objects that appear in applied mathematics: complex numbers, square matrices with entries in a field, real or complex valued functions defined on a set, polynomials and power series with coefficients in a field, integers with modular arithmetic, etc. Ring theory deals with such objects. A ring R is an abelian group with a multiplication operation $(a, b) \mapsto ab$ which is associative and satisfies the distributive laws $a(b + c) = ab + ab$ and $(a + b)c = ac + bc$. The multiplication may or may not be commutative. This distinction yields two quite different theories: the theory of respectively commutative or non-commutative rings. Commutative ring theory originated in algebraic number theory, algebraic geometry, and invariant theory. Noncommutative ring theory began with attempts to extend the complex numbers to various hyper-complex number systems. The genesis of the theories of commutative and noncommutative rings dates back to the early 19th century.

Rough set theory, proposed in 1982 by Pawlak [7], is in a state of constant development. Its methodology is concerned with the classification and analysis of imprecise, uncertain or incomplete information and knowledge, and of is considered one of the first non-statistical approaches in data analysis. The theory has found applications in many domains, such as decision support, engineering, environment, banking, medicine and others. The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in Pawlak rough set model is an

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equivalence relation. In Pawlak rough sets, the equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. Lin [8] proposed a more general framework for the study of approximation operators by using the so-called neighborhood systems from a topological space. In a neighborhood system, each element of a universe is associated with a family of subsets of the universe. This family is called a neighborhood system of the element, and each member in the family is called a neighborhood of the element. Any subset of the universe can be approximated based on neighborhood systems of all elements in the universe, also see [9]. Davvaz et al [1, 2, 3, 5, 6, 10, 11] studied relationships between ring theory and rough sets. In [4], Davvaz considered the relation γ and its transitive closure γ^* on a semigroup. The relation γ is the smallest equivalence relation on S so that S/γ^* is a commutative semigroup. Based on the relation γ, he defined a neighborhood system for each element of S, and presented a general framework of the study of approximations in semigroups.

2. α-relation on rings

Definition 2.1. *Let R be a ring. A congruence relation ρ on R is an equivalence relation that satisfy*

$$r_1 + s_1 \ \rho \ r_2 + s_2 \ \ \text{and} \ \ r_1 s_1 \ \rho \ r_2 s_2,$$

whenever $r_1 \rho r_2$ and $s_1 \rho s_2$.

For a congruence on a ring, the equivalence class containing 0 is always a two-sided ideal, and the two operations on the set of equivalence classes define the corresponding quotient ring.

Lemma 2.1. *Let R be a ring and ρ be an equivalence relation on R. Then, ρ is a congruence relation on R if and only if for every $x, y, a \in R$,*

$$x\rho y \ \Rightarrow \ \begin{cases} x + a \ \rho \ y + a, & a + x \ \rho \ a + y, \\ x \cdot a \ \rho \ y \cdot a, & a \cdot x \ \rho \ a \cdot y. \end{cases}$$

Proof. It is straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 2.2. *Let R be a (non-commutative) ring. We define the relation α as follows:*
$x \ \alpha \ y \iff \exists n \in \mathbb{N}, \ \exists(k_1, \ldots, k_n) \in \mathbb{N}^n$ *and* $[\exists(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, \ \exists \sigma_i \in \mathbb{S}_{k_i}, (i = 1, \ldots, n)]$ *such that*

$$x = \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right) \quad \text{and} \quad y = \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{i\sigma_i(j)}\right).$$

The relation α is reflexive and symmetric. Let α^* be the transitive closure of α.

Theorem 2.1. *α^* is a congruence relation on R.*

Proof. If $x\alpha y$, then $\exists n \in \mathbb{N}$, $\exists (k_1, \ldots, k_n) \in \mathbb{N}^n$, and $[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$, $\exists \sigma_i \in \mathbb{S}_{k_i}$, $(i = 1, \ldots, n)]$ such that

$$x = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{i\sigma_i(j)}).$$

and so

$$x + a = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) + a \quad \text{and} \quad y + a = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{i\sigma_i(j)}) + a.$$

Now, let $k_{n+1} = 1$, $x_{n+1\ 1} = a$, $\sigma_{n+1} = id$. Thus,

$$x + a = \sum_{i=1}^{n+1}(\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y + a = \sum_{i=1}^{n+1}(\prod_{j=1}^{k_i} x_{i\sigma_i(j)}).$$

Therefore, $x + a\ \alpha\ y + a$. In the same way, we can show that $a + x\ \alpha\ a + y$. Now, it is easy to see that

$$x + a\ \alpha^*\ y + a \quad \text{and} \quad a + x\ \alpha^*\ a + y.$$

Now, note that

$$xa = \big(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})\big)a \quad \text{and} \quad ya = \big(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{i\sigma_i(j)})\big)a,$$

which yields that

$$xa = \sum_{i=1}^{n}\big((\prod_{j=1}^{k_i} x_{ij})a\big) \quad \text{and} \quad ya = \sum_{i=1}^{n}\big((\prod_{j=1}^{k_i} x_{i\sigma_i(j)})a\big).$$

We put $k_i' = k_i + 1$, $x_{ik_i'} = a$ and define $\tau_i(r) = \sigma_i(r)$ for all $r = 1, \ldots, k_i$ and $\tau_i(k_i + 1) = k_i + 1$. In this case, $\tau_i \in \mathbb{S}_{k_i'}$ $(i = 1, \ldots, n)$. Thus,

$$xa = \sum_{i=1}^{n}(\prod_{j=1}^{k_i'} x_{ij}) \quad \text{and} \quad ya = \sum_{i=1}^{n}(\prod_{j=1}^{k_i'} x_{i\tau_i(j)}).$$

Therefore, $xa\ \alpha\ ya$ and so $xa\ \alpha^*\ ya$. Similarly, we obtain $ax\ \alpha^*\ ay$. This completes the proof. □

We define $\oplus$ and $\odot$ on R/α^* in the usual manner:

$$\alpha^*(a) \oplus \alpha^*(b) = \alpha^*(a + b),$$
$$\alpha^*(a) \odot \alpha^*(b) = \alpha^*(ab).$$

Corollary 2.1. *The quotient R/α^* is a commutative ring.*

Proof. Since α^* is a congruence relation, R/α^* is a ring. Suppose that σ is the permutation of $\mathbb{S}_2$ such that $\sigma(1) = 2$. Clearly, we have $x_1 x_2\ \alpha\ x_{\sigma(1)} x_{\sigma(2)}$. Then, $x_1 x_2\ \alpha^*\ x_{\sigma(1)} x_{\sigma(2)}$. Therefore, R/α^* is a commutative ring. □

Theorem 2.2. *The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring.*

Proof. Let θ be an equivalence relation such that R/θ is a commutative ring and let $\varphi : R \longrightarrow R/\theta$ be the canonical projection. If $x\alpha y$, then there exist $n \in \mathbb{N}$, $(k_1, \ldots, k_n) \in \mathbb{N}^n$ and there exist $(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$ and $\sigma_i \in \mathbb{S}_{k_i}$ $(i = 1, \ldots, n)$ such that

$$x = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{i\sigma_i(j)}).$$

Hence,

$$\varphi(x) = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} \varphi(x_{ij})) \quad \text{and} \quad \varphi(y) = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} \varphi(x_{i\sigma_i(j)})).$$

By the commutativity of R/θ, it follows that $\varphi(x) = \varphi(y)$. Thus $x\alpha y$ implies that $x\theta y$. Finally, let $x\alpha^* y$. Then, there exist $z_1, \ldots, z_m \in R$ such that $x = z_1\alpha z_2$, $z_2\alpha z_3$, $\ldots$, $z_{n-1}\alpha z_n = y$, and so $x = z_1\theta z_2$, $z_2\theta z_3$, $\ldots$, $z_{n-1}\theta z_n = y$

Since θ is transitively closed, we obtain $x\theta y$. Hence,

$$x \in \alpha^*(y) \implies x \in \theta(y).$$

Therefore $\alpha^* \subseteq \theta$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Neighborhood operators

Definition 3.1. *For the relation α on R and a positive integer k, we now define a notion of binary relation α^k called the k-step-relation of α as follows:*

(1) $\alpha^1 = \alpha$,
(2) $\alpha^k = \{(x, y) \in R \times R \mid \text{there exist } y_1, y_2, \ldots, y_i \in R, \ 1 \le i \le k-1, \text{ such that } x\alpha y_1, \ y_1\alpha y_2, \ldots, y_i\alpha y\} \cup \alpha^1, \ k \ge 2.$

It is easy to see that

$$\alpha^{k+1} = \alpha^k \cup \{ \ (x,y) \in R \times R \mid \text{there exist } y_1, \ldots, y_k \in R, \text{such that } x\alpha y_1, \ y_1\alpha y_2, \ldots, y_k\alpha y\}.$$

Obviously, $\alpha^k \subseteq \alpha^{k+1}$, and there exists $n \in \mathbb{N}$ such that $\alpha^k = \alpha^n$ for all $k \ge n$. (In fact $\alpha^n = \alpha^*$ is nothing else but the transitive closure of α). Of course α^* is transitive. The relation α^k can be conveniently expressed as a mapping from R to $\wp(R)$, $N_k(x) = \{y \in R \mid x\alpha^k y\}$ by collecting all α^k-related elements for each element $x \in R$. The set $N_k(x)$ may be viewed as a α^k-*neighborhood* of x defined by the binary relation α^k.

Based on the relation α^k on R, we can obtain a *neighborhood system* for each element x: $\{N_k(x) \mid k \ge 1\}$. This neighborhood system is monotonically increasing with respect to k. We can also observe that

$$N_k(x) = \{y \in R \mid \ \text{there exist } y_1, y_2, \ldots, y_i \in R \text{ such that } x\alpha y_1, \ y_1\alpha y_2, \ldots, y_i\alpha y, \ 1 \le i \le k-1, \text{ or } x\alpha^k y\}.$$

If A and B are non-empty subsets of a ring R, then $A + B = \{a + b \mid a \in A, b \in B\}$ and AB denote the set of all finite sums $\{a_1b_1 + a_2b_2 + \ldots + a_nb_n \mid n \in \mathbb{N}, \ a_i \in A, \ b_i \in B\}$.

Theorem 3.1. *For each $a, b \in R$ and natural numbers k, l we have*

$$N_k(a) + N_l(b) \subseteq N_{k+l-1}(a + b).$$

Proof. Suppose that $x \in N_k(a) + N_l(b)$. Then there exist $a' \in N_k(a)$ and $b' \in N_l(b)$ such that $x = a' + b'$. Since $a' \in N_k(a)$, then $a'\alpha^k a$ and so there exist $\{x_1, \ldots, x_{k+1}\} \subseteq R$ with $x_1 = a'$, $x_{k+1} = a$ such that $x_1 \; \alpha \; x_2$, $x_2 \; \alpha \; x_3$, ..., $x_k \; \alpha \; x_{k+1}$. Hence, for $t = 1, \ldots, k$,

$$x_t \; \alpha \; x_{t+1} \iff \exists n_t \in \mathbb{N}, \; \exists(h_{t1}, \ldots, h_{tn_t}) \in \mathbb{N}^{n_t} \text{ and } [\exists(u_{ti1}, \ldots, u_{tih_{t_i}}) \in R^{h_{ti}},$$

$\exists \sigma_i \in \mathbb{S}_{h_{ti}}, \; (i = 1, \ldots, n_t)]$ such that

$$x_t = \sum_{i=1}^{n_t}(\prod_{j=1}^{h_{ti}} u_{tij}) \quad \text{and} \quad x_{t+1} = \sum_{i=1}^{n_t}(\prod_{j=1}^{h_{ti}} u_{ti\sigma_i(j)}).$$

Also, since $b' \in N_l(b)$, then $b' \; \alpha^l \; b$ and so there exist $\{y_1, \ldots, y_{l+1}\} \subseteq R$ with $y_1 = b'$, $y_{l+1} = b$ such that $y_1 \; \alpha \; y_2$, $y_2 \; \alpha \; y_3$, ..., $y_l \; \alpha \; y_{l+1}$. Hence, for $s = 1, \ldots, l$,

$$y_s \; \alpha \; y_{s+1} \iff \exists m_s \in \mathbb{N}, \; \exists(h'_{s1}, \ldots, h'_{sm_s}) \in \mathbb{N}^{m_s} \text{ and } [\exists(v_{si1}, \ldots, v_{sih'_{s_i}}) \in R^{h'_{si}},$$

$\exists \sigma_i \in \mathbb{S}_{h'_{si}}, \; (i = 1, \ldots, m_s)]$ such that

$$y_s = \sum_{i=1}^{m_s}(\prod_{j=1}^{h'_{si}} v_{sij}) \quad \text{and} \quad y_{s+1} = \sum_{i=1}^{m_s}(\prod_{j=1}^{h'_{si}} v_{si\sigma_i(j)}).$$

Therefore, we obtain

$$x_t + y_1 = \sum_{i=1}^{n_t}(\prod_{j=1}^{h_{ti}} u_{tij}) + \sum_{i=1}^{m_1}(\prod_{j=1}^{h'_{1i}} v_{1ij}),$$

$$x_{t+1} + y_1 = \sum_{i=1}^{n_t}(\prod_{j=1}^{h_{ti}} u_{ti\sigma_i(j)}) + \sum_{i=1}^{m_1}(\prod_{j=1}^{h'_{1i}} v_{1ij}),$$

and

$$x_{k+1} + y_s = \sum_{i=1}^{n_k}(\prod_{j=1}^{h_{ki}} u_{ki\sigma_i(j)}) + \sum_{i=1}^{m_s}(\prod_{j=1}^{h'_{si}} v_{sij}),$$

$$x_{k+1} + y_{s+1} = \sum_{i=1}^{n_k}(\prod_{j=1}^{h_{ki}} u_{ki\sigma_i(j)}) + \sum_{i=1}^{m_s}(\prod_{j=1}^{h'_{si}} v_{si\sigma_i(j)}).$$

If we pick up elements $z_1, \ldots, z_{k+l}$ such that

$$z_i = x_i + y_1, \qquad i = 1, \ldots, k,$$
$$z_{k+j} = x_{k+1} + y_{j+1}, \quad j = 1, \ldots, l.$$

Then, $z_1 \; \alpha^{k+l-1} \; z_{m+1}$. So $x = a' + b' = x_1 + y_1 \; \alpha^{k+l-1} \; x_{k+1} + y_{l+1} = a + b$. Therefore $x \in N_{k+l-1}(a+b)$. □

For a neighborhood operator N_k on R, we can extend N_k from $\wp(R)$ to $\wp(R)$ by: $N_k(X) = \bigcup_{x \in X} N_k(x)$ for all $X \subseteq R$. So, we can directly deduce that

Proposition 3.1. *We have*

(1) $A \subseteq B \implies N_k(A) \subseteq N_k(B)$,
(2) *for all* $k, l \geq 1$, *we have* $N_l(N_k(x)) \subseteq N_{l+k}(x)$.

4. Approximation operators

If θ^* is a congruence relation on R such that R/θ^* is a commutative ring, then $\alpha^* \subseteq \theta^*$.

Let R be a ring and A be a non-empty subset of R. We define the *lower and upper approximations* of A with respect to α^* as follows:

$$\underline{\alpha^*}(A) = \{x \in S \mid \alpha^*(x) \subseteq A\} \text{ and } \overline{\alpha^*(A)} = \{x \in S \mid \alpha^*(x) \cap A \neq \emptyset\}.$$

Similarly, we can define the lower and upper approximations of A with respect to η^*. In this case, we have

$$\underline{\theta^*(A)} \subseteq \underline{\alpha^*(A)} \subseteq A \subseteq \overline{\alpha^*(A)} \subseteq \overline{\theta^*(A)}.$$

In [1], Davvaz gave some properties of the lower and upper approximations with respect to an ideal of a ring, also see [2, 3, 6]. Since α^* is a congruence relation, all the results in [1] are true for the relation α^*.

Definition 4.1. *For the relation α, by substituting equivalence class $\alpha^*(x)$ with α^k-neighborhood $N_k(x)$ in the previous definition, we can define a pair of lower and upper approximation operators with respect to N_k as follows:*

$$\underline{apr}_k(A) = \{x \in R \mid N_k(x) \subseteq A\} \text{ and } \overline{apr}_k(A) = \{x \in R \mid N_k(x) \cap A \neq \emptyset\}.$$

The set $\underline{apr}_k(A)$ consists of those elements whose α^k-neighborhoods are contained in A, and $\overline{apr}_k(A)$ consists of those elements whose α^k-neighborhoods have a non-empty intersection with A.

Proposition 4.1. *If A is a non-empty subset of R, then we have*

(1) $\underline{apr}_{k+1}(A) \subseteq \underline{apr}_k(A)$,
(2) $\overline{apr}_k(A) \subseteq \overline{apr}_{k+1}(A)$.

Therefore:

Corollary 4.1. *We have*

$$\bigcup\{x \mid x \in \underline{\alpha^*(A)}\} = \bigcap_k \underline{apr}_k(A) \text{ and } \bigcup\{x \mid x \in \overline{\alpha^*(A)}\} = \bigcup_k \overline{apr}_k(A).$$

Proposition 4.2. *If A and B are non-empty subsets of R, then the pair of approximation operators satisfies the following properties:*

(1) $\underline{apr}_k(A) \subseteq A \subseteq \overline{apr}_k(A)$,
(2) $\underline{apr}_k(A) = (\overline{apr}_k(A^c))^c$,
(3) $\overline{apr}_k(A) = (\underline{apr}_k(A^c))^c$,
(4) $\underline{apr}_k(A \cap B) = \underline{apr}_k(A) \cap \underline{apr}_k(B)$,
(5) $\overline{apr}_k(A \cup B) = \overline{apr}_k(A) \cup \overline{apr}_k(B)$,
(6) $\underline{apr}_k(A \cup B) \supseteq \underline{apr}_k(A) \cup \underline{apr}_k(B)$,
(7) $\overline{apr}_k(A \cap B) \subseteq \overline{apr}_k(A) \cap \overline{apr}_k(B)$,
(8) $A \subseteq B \implies \underline{apr}_k(A) \subseteq \underline{apr}_k(B)$,
(9) $A \subseteq B \implies \overline{apr}_k(A) \subseteq \overline{apr}_k(B)$.

Proof. It is straightforward. □

Theorem 4.1. *Let A be a non-empty subset of R. For all $k \geq l \geq 1$, we have*

(1) $A \subseteq \underline{apr}_l(\overline{apr}_k(A))$,

(2) $\overline{apr}_l(\underline{apr}_k(A)) \subseteq A$.

Proof. (1) Suppose that $a \in A$. If $N_l(a) = \emptyset$. Then it is clear that $N_l(a) \subseteq \overline{apr}_k(A)$, which implies that $a \in \underline{apr}_l(\overline{apr}_k(A))$, and so $A \subseteq \underline{apr}_l(\overline{apr}_k(A))$. If $N_l(a) \neq \emptyset$, then for each $b \in N_l(a)$, we have $a \in N_l(b)$. Hence, $N_l(b) \cap A \neq \emptyset$. Now, we have $b \in \overline{apr}_l(A)$, and by Proposition 4.1, we obtain $b \in \overline{apr}_k(A)$. Therefore $N_l(a) \subseteq \overline{apr}_k(A)$, which implies that $a \in \underline{apr}_l(\overline{apr}_k(A))$, and so $A \subseteq \underline{apr}_l(\overline{apr}_k(A))$.

(2) Suppose that $a \in \overline{apr}_l(\underline{apr}_k(A))$. Then, we have $N_l(a) \cap \underline{apr}_k(A) \neq \emptyset$, and so there exists $b \in N_l(a) \cap \underline{apr}_k(A)$. Therefore $a \in N_l(b)$ and $N_k(b) \subseteq A$. Hence $a \in N_l(b) \subseteq N_k(b) \subseteq A$, and so we conclude that $\overline{apr}_l(\underline{apr}_k(A)) \subseteq A$. $\qquad\square$

Theorem 4.2. *For all $k, l \geq 1$ and $A \subseteq R$, we have*

(1) $\underline{apr}_{l+k}(A) \subseteq \underline{apr}_l(\underline{apr}_k(A))$,

(2) $\overline{apr}_{l+k}(A) \supseteq \overline{apr}_l(\overline{apr}_k(A))$.

Proof. (1) Suppose that $a \in \underline{apr}_{l+k}(A)$. Then $N_{l+k}(a) \subseteq A$. Using Proposition 3.1, we have $N_k(N_l(a)) \subseteq N_{k+l}(a) \subseteq A$, which implies that $N_l(a) \subseteq \underline{apr}_k(A)$. Therefore, $a \in \underline{apr}_l(\underline{apr}_k(A))$.

(2) Suppose that $a \in \overline{apr}_l(\overline{apr}_k(A))$. Then $N_l(a) \cap \overline{apr}_k(A) \neq \emptyset$, and so there exists $b \in N_l(a) \cap \overline{apr}_k(A)$. Since $b \in \overline{apr}_k(A)$, then $N_k(b) \cap A \neq \emptyset$. Now, we have

$$\emptyset \neq N_k(b) \cap A \subseteq N_k(N_l(a)) \cap A \subseteq N_{l+k}(a) \cap A,$$

and so $N_{l+k}(a) \cap A \neq \emptyset$, which implies that $a \in \overline{apr}_{l+k}(A)$. $\qquad\square$

Theorem 4.3. *If A, B are non-empty subsets of R, then*

$$\overline{apr}_k(A) + \overline{apr}_l(B) \subseteq \overline{apr}_{k+l-1}(A + B).$$

Proof. Suppose that z be any element of $\overline{apr}_k(A) + \overline{apr}_l(B)$. Then, there exist $x \in \overline{apr}_k(A)$ and $y \in \overline{apr}_l(B)$ such that $z = x + y$. Since $x \in \overline{apr}_k(A)$ and $y \in \overline{apr}_l(B)$, then there exist $a, b \in R$ such that $a \in N_k(x) \cap A$ and $b \in N_l(y) \cap B$. So, $a \in N_k(x)$ and $b \in N_l(y)$. By Theorem 3.1, we have $N_k(x) + N_l(y) \subseteq N_{k+l-1}(z)$. Since $a+b \in A+B$, we obtain $a+b \in N_{k+l-1}(z) \cap A+B$, and so $z \in \overline{apr}_{k+l-1}(A+B)$. This completes the proof. $\qquad\square$

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