FIXED POINT APPROACH THROUGH SIMULATION FUNCTION AND ASYMPTOTIC REGULARITY

Kushal Roy[1]

In this paper, we have introduced a new contractive mapping known as $\mathcal{Z}$-Proinov-Górnicki contractive mapping as a generalization of Kannan contractive mapping. Also some fixed point and common fixed point theorems have been proved for such type of mapping in the setting of metric space via asymptotic regularity and weak continuity. Our results extend and improve some theorems of other researchers available in several literatures. Finally this paper has been furnished by some examples that support our proven theorems.

Keywords: Fixed point, $\mathcal{Z}$-Proinov-Górnicki contractive mapping, metric space, asymptotic regularity, orbital continuity and k-continuity.
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1. Introduction and Preliminaries

The definition of simulation function was given by Khojasteh et al. (see [7]). Such functions have been used in the contractive type mapping known as $\mathcal{Z}-$contraction introduced by Khojasteh et al. in the year 2015.

Definition 1.1. [7] *A function $\zeta : [0, \infty)^2 \to \mathbb{R}$ is called simulation function, if it satisfies the following conditions:*

(ζ_1) $\zeta(0, 0) = 0$,

(ζ_2) $\zeta(t, s) < s - t$ *for all* $s, t > 0$,

(ζ_3) *If* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then* $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Here we give an example of simulation function.

Example 1.1. [7] *(i) Let $\zeta_1 : [0, \infty)^2 \to \mathbb{R}$ be defined by*

$$\zeta_1(t, s) = \frac{s}{s + 1} - t \text{ for all } t, s \in [0, \infty). \tag{1}$$

(ii) Also let $\zeta_2 : [0, \infty)^2 \to \mathbb{R}$ be given by

$$\zeta_2(t, s) = s - \varphi(s) - t \text{ for all } t, s \in [0, \infty), \tag{2}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$. Then ζ_1, ζ_2 are simulation functions.

Another examples of simulation functions can be found in [4], [12].

[1]Assistant Professor, Department of Mathematics, Supreme Knowledge Foundation Group of Institutions, Chandannagar, Hooghly-712139, West Bengal, India, e-mail: kushal.roy93@gmail.com

Definition 1.2. [7] *A self mapping T on a metric space (X, d) is said to be $\mathcal{Z}-$contraction if there exists a simulation function ζ such that for all $x, y \in X$, $\zeta(d(Tx, Ty), d(x, y)) \geq 0$.*

Theorem 1.1. [7] *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}-$contraction with respect to some simulation function ζ. Then T has a unique fixed point in X.*

Recently Radenović et al. [12] have extended and improved some results on simulation functions established by several authors. By using the following Lemma they have got much shorter proofs than the corresponding ones given in the literature.

Lemma 1.1. [12] *Let $\{x_n\}$ be a sequence in a metric space (X, d) such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence in X then there exists $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and such that the following sequences*

$$\{d(x_{m_k}, x_{n_k})\}, \{d(x_{m_k}, x_{n_k+1})\}, \{d(x_{m_k-1}, x_{n_k})\}, \{d(x_{m_k-1}, x_{n_k+1})\}, \{d(x_{m_k+1}, x_{n_k+1})\} \quad (3)$$

converge to ϵ as $k \to \infty$.

Recently J. Górnicki [9] studied a new class of contractive mappings and proved a fixed point theorem for such mappings over metric spaces with assumption of continuity, which is given as follows. It is to be noted that asymptotic regularity [2, 1] has been assumed for these mappings in all over the metric space.

Theorem 1.2. [9] *In a complete metric space (X, d) a continuous and asymptotically regular map $T : X \to X$ satisfying*

$$d(Tx, Ty) \leq \alpha d(x, y) + K \{d(x, Tx) + d(y, Ty)\} \text{ for all } x, y \in X \quad (4)$$

for some $\alpha \in [0, 1)$ and for some $K \geq 0$ has a unique fixed point $u \in X$ and for each $x \in X$, $T^n x \to u$ as $n \to \infty$.

Bisht [3] replaced the assumption of continuity in Theorem 1.2 by a weaker version of continuity condition like orbital continuity or k-continuity (For definitions of orbital continuity and k-continuity one may refer [3]).

Panja et al. [10] have generalized the contractive condition (4) and have introduced a new type of contractive mapping called Ćirić-Proinov-Górnicki type mapping.

Let us consider the class $\mathcal{F}$ of all functions $F : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(i) $F(0, 0) = 0$, (ii) F is continuous at $(0, 0)$.

Definition 1.3. [10] *In a metric space (X, d) a mapping $T : X \to X$ is said to be Ćirić-Proinov-Górnicki type mapping if there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Ty), d(y, Tx)\} + F(d(x, Tx), d(y, Ty)) \quad (5)$$

for all $x, y \in X$ and for some $F \in \mathcal{F}$.

Theorem 1.3. [10] *Let (X, d) be a complete metric space and $T : X \to X$ be asymptotically regular Ćirić-Proinov-Górnicki type mapping. Then T has a unique fixed point provided either T is k-continuous for $k \geq 1$ or T is orbitally continuous.*

Definition 1.4. *In a metric space (X, d), let $T, S : X \to X$ be two mappings. Then,*

(i) T is said to be asymptotic regular with respect to S at a point $x_0 \in X$ [15] if there exists a sequence $\{x_n\}_{n=0,1,\cdots}$ in X such that $Tx_n = Sx_{n+1}$ for all $n = 0, 1, \cdots$ and $d(Sx_{n+1}, Sx_{n+2}) \to 0$ as $n \to \infty$.

(ii) Let $\{x_n\}$ be a sequence in X such that $Tx_n = Sx_{n+1}$ for all $n = 0, 1, \cdots$ and $Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then T (resp. S) is said to be (T, S)-orbitally continuous [11] if $TTx_n \to Tz$ as $n \to \infty$ (resp. $STx_n \to Sz$ as $n \to \infty$).

Definition 1.5. [5] *In a metric space (X, d), two maps $T, S : X \to X$ are said to be compatible if $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$, whenever $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = u$, for some $u \in X$.*

Roy et al. [14] have recently proved two common fixed point theorems using the contractive condition (5) which are given below.

Theorem 1.4. [14] *Let (X, d) be a complete metric space and $T, S : X \to X$ be two asymptotically regular mappings satisfying*

$$d(Tx, Sy) \leq \lambda \max\{d(x, y), d(x, Sy), d(y, Tx)\} + F(d(x, Tx), d(y, Sy)) \tag{6}$$

for all $x, y \in X$, for some $\lambda \in [0, 1)$ and for some $F \in \mathcal{F}$. Then T and S have a unique common fixed point in X, provided T and S are either k-continuous for some $k \geq 1$ or orbitally continuous.

Theorem 1.5. [14] *Let $X, d)$ be a complete metric space and $T, S : X \to X$ be two mappings such that T is asymptotic regular with respect to S at a point $x_0 \in X$ and satisfy the following condition*

$$d(Tx, Ty) \leq \lambda \max\{d(Sx, Sy), d(Tx, Sy), d(Sx, Ty)\} + F(d(Tx, Sx), d(Ty, Sy)) \tag{7}$$

for all $x, y \in X$, for some $\lambda \in [0, 1)$ and for some $F \in \mathcal{F}$. Then T and S have a unique common fixed point in X, provided T and S are (T, S)-orbitally continuous and compatible.

2. Some fixed point and common fixed point theorems for Górnicki contractive type mappings

In this section we introduce a new generalized Kannan type contractive mapping with the help of simulation functions. In the following we denote the set of all simulations functions as $\mathfrak{S}$.

Definition 2.1. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be $\mathcal{Z}$-Proinov-Górnicki contractive mapping if*

$$\zeta(d(Tx, Ty), d(x, y)) + K[d(x, Tx) + d(y, Ty)] \geq 0 \text{ for all } x, y \in X, \tag{8}$$

where $\zeta \in \mathfrak{S}$ and $K \geq 0$.

Example 2.1. *(a) Let us consider $X = [0, +\infty)$ endowed with the usual metric of reals. Also let $T : X \to X$ be given by*

$$Tx = \frac{x}{x+1} \text{ for all } x \in X. \tag{9}$$

Then T is a $\mathcal{Z}$-Proinov-Górnicki contractive mapping for the simulation function ζ_1 defined by $\zeta_1(t, s) = \frac{s}{s+1} - t$ for all $t, s \geq 0$ but not usual Górnicki type contractive mapping.

(b) Let us consider $X = [0, +\infty)$ endowed with the usual metric of reals and $T : X \to X$ be given by

$$Tx = \frac{x}{x^2+1} \text{ for all } x \in X. \tag{10}$$

Then T is a $\mathcal{Z}$-Proinov-Górnicki contractive mapping for the simulation function ζ_3 defined by $\zeta_3(t,s) = \frac{s}{\sqrt{s^2+1}} - t$ for all $t, s \in [0, +\infty)$ but not usual Górnicki type contractive mapping.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-Proinov-Górnicki contractive mapping. If T is asymptotically regular and additionally k-continuous or orbitally continuous then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be chosen as arbitrary and consider the Picard iterating sequence $\{x_n\}$ in X defined by $x_n = T^n x_0$ for all $n \geq 1$. Since T is asymptotically regular it follows that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not Cauchy then by Lemma 1.1 we see that there exists $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences $\{d(x_{m_k}, x_{n_k})\}, \{d(x_{m_k}, x_{n_k+1})\}, \{d(x_{m_k-1}, x_{n_k})\}$, $\{d(x_{m_k-1}, x_{n_k+1})\}, \{d(x_{m_k+1}, x_{n_k+1})\}$ converge to ϵ as $k \to \infty$. Now from the contractive condition (8) we have

$$\zeta(d(x_{m_k+1}, x_{n_k+1}), d(x_{m_k}, x_{n_k})) + K[d(x_{m_k}, x_{m_k+1}) + d(x_{n_k}, x_{n_k+1})] \geq 0 \quad \text{for all } k \in \mathbb{N}. \tag{11}$$

Taking $k \to \infty$ we get,

$$\lim_{k\to\infty} \zeta(d(x_{m_k+1}, x_{n_k+1}), d(x_{m_k}, x_{n_k})) \geq 0. \tag{12}$$

Now since $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon = \lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1})$. So by the property (ζ_3) of ζ we obtain

$$\lim_{k\to\infty} \zeta(d(x_{m_k+1}, x_{n_k+1}), d(x_{m_k}, x_{n_k})) < 0, \tag{13}$$

which leads us to a contradiction. Therefore $\{x_n\}$ must be Cauchy sequence in X. By the completeness of X it follows that $\{x_n\}$ converges to some $u \in X$. Since T is either k-continuous or orbitally continuous in X, implies that $Tu = u$. Let u and v be two fixed points of T. Then

$$\zeta(d(Tu, Tv), d(u, v)) + K[d(u, Tu) + d(v, Tv)] \geq 0, \tag{14}$$

which implies that $\zeta(d(u, v), d(u, v)) \geq 0$. If $u \neq v$ then by the property (ζ_2) of ζ we have

$$0 \leq \zeta(d(u, v), d(u, v)) < d(u, v) - d(u, v) = 0, \text{ which is not possible.} \tag{15}$$

Hence T has a unique fixed point in X. □

Theorem 2.2. *Let (X, d) be a complete metric space and $T, F : X \to X$ be two mappings such that T is asymptotic regular with respect to F at $x_0 \in X$ and satisfy the following condition*

$$\zeta(d(Tx, Ty), d(Fx, Fy)) + K[d(Fx, Tx) + d(Fy, Ty)] \geq 0 \text{ for all } x, y \in X, \tag{16}$$

for all $x, y \in X$, for some $K \geq 0$ and for some $\zeta \in \mathfrak{S}$. Then T and F have a unique common fixed point provided T and F are (T, F)-orbitally continuous and compatible.

Proof. Since T is asymptotic regular with respect to F at $x_0 \in X$, so there exists a sequence $\{x_n\}$ in X such that $Tx_n = Fx_{n+1} = h_n$ (say) for all $n = 0, 1, 2, \cdots$ and $d(Fx_{n+1}, Fx_{n+2}) \to 0$ as $n \to \infty$, i.e., $d(h_n, h_{n+1}) \to 0$ as $n \to \infty$.

First we will show that h_n is a Cauchy sequence in X. If $\{h_n\}$ is not Cauchy then by Lemma 1.1 we see that there exists $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences $\{d(h_{m_k}, h_{n_k})\}$, $\{d(h_{m_k}, h_{n_k+1})\}, \{d(h_{m_k-1}, h_{n_k})\}, \{d(h_{m_k-1}, h_{n_k+1})\}, \{d(h_{m_k+1}, h_{n_k+1})\}$ converge to ϵ as $k \to \infty$. From the contractive condition (16) we have for all $k \in \mathbb{N}$

$$\begin{aligned} &\zeta(d(Tx_{m_k+1}, Tx_{n_k+1}), d(Fx_{m_k+1}, Fx_{n_k+1}))+ \\ &K[d(Fx_{m_k+1}, Tx_{m_k+1}) + d(Fx_{n_k+1}, Tx_{n_k+1})] \geq 0 \\ \Rightarrow\ &\zeta(d(h_{m_k+1}, h_{n_k+1}), d(h_{m_k}, h_{n_k})) + K[d(h_{m_k}, h_{m_k+1}) + d(h_{n_k}, h_{n_k+1})] \geq 0. \end{aligned} \tag{17}$$

Taking $k \to \infty$ we see that

$$\lim_{k\to\infty} \zeta(d(h_{m_k+1}, h_{n_k+1}), d(h_{m_k}, h_{n_k})) \geq 0, \text{ a contradiction.} \tag{18}$$

Therefore $\{h_n\}$ is Cauchy and completeness of X implies that there exists $z \in X$ such that $h_n \to z$ as $n \to \infty$, i.e. $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Fx_{n+1} = z$.

Since T and F are (T, F)-orbitally continuous, it follows that $\lim_{n\to\infty} T^2x_{n-1} = \lim_{n\to\infty} TFx_n = Tz$ and $\lim_{n\to\infty} F^2x_{n+1} = \lim_{n\to\infty} FTx_n = Fz$. Due to the compatibility of T and F it is seen that $Tz = Fz$ and $T^2z = T(Fz) = F(Tz) = F^2z$. Therefore using the contractive condition (16) we get

$$\begin{aligned} &\zeta(d(Tz, T^2z), d(Fz, F(Tz))) + K[d(Fz, Tz) + d(F(Tz), T^2z)] \geq 0 \\ &\Rightarrow \zeta(d(Tz, T^2z), d(Tz, T^2z)) \geq 0. \end{aligned} \tag{19}$$

If $T^2z \neq Tz$ then from the property of ζ we see that

$$0 \leq \zeta(d(Tz, T^2z), d(Tz, T^2z)) < d(Tz, T^2z) - d(Tz, T^2z) = 0, \text{ a contradiction.} \tag{20}$$

Hence $T^2z = F(Tz) = Tz$ and T and F have a common fixed point in X. Uniqueness of common fixed point of T and F can be proved in a similar way as in Theorem 2.3. □

Theorem 2.3. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings satisfying*

$$\zeta(d(Tx, Sy), d(x, y)) + K[d(x, Tx) + d(y, Sy)] \geq 0 \text{ for all } x, y \in X, \tag{21}$$

where $\zeta \in \mathfrak{S}$ and $K \geq 0$. If T and S both are asymptotically regular and $\{d(T^nx_0, S^nx_0)\}$ is convergent for some $x_0 \in X$, then T and S have a unique common fixed point in X, provided T, S are either k-continuous or orbitally continuous in X.

Proof. First we show that $\lim_{n\to\infty} d(T^nx_0, S^nx_0) = 0$. If there exists some $N \geq 1$ such that $T^nx_0 = S^nx_0$ for all $n \geq N$ then clearly $\lim_{n\to\infty} d(T^nx_0, S^nx_0) = 0$. So let $\{n_i\} \subset \mathbb{N}$ be such that $T^{n_i}x_0 \neq S^{n_i}x_0$ for all $i \geq 1$. If $\lim_{i\to\infty} d(T^{n_i}x_0, S^{n_i}x_0) = 0$ then we have nothing to prove. So let us assume that $\lim_{i\to\infty} d(T^{n_i}x_0, S^{n_i}x_0) = \epsilon > 0$. Now since $\{d(T^nx_0, S^nx_0)\}$ is convergent then $d(T^nx_0, S^nx_0) \to \epsilon$ as $n \to \infty$.

Therefore $\lim_{i\to\infty} d(T^{n_i-1}x_0, S^{n_i-1}x_0) = \epsilon$ and by using the contractive condition (21) we see that

$$\begin{aligned}&\zeta(d(T^{n_i}x_0, S^{n_i}x_0), d(T^{n_i-1}x_0, S^{n_i-1}x_0))+\\&K[d(T^{n_i-1}x_0, T^{n_i}x_0) + d(S^{n_i-1}x_0, S^{n_i}x_0)] \geq 0 \text{ for all } i \in \mathbb{N}\\\Rightarrow &\limsup_{i\to\infty} \zeta(d(T^{n_i}x_0, S^{n_i}x_0), d(T^{n_i-1}x_0, S^{n_i-1}x_0)) \geq 0, \text{ a contradiction.}\end{aligned} \tag{22}$$

Hence we have $\lim_{n\to\infty} d(T^n x_0, S^n x_0) = 0$. Let $x_n = T^n x_0$ for all $n \geq 1$. Now we prove that $\{x_n\}$ is Cauchy in X. If not then by Lemma 1.1 we see that there exists $\delta > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences $\{d(x_{m_k}, x_{n_k})\}, \{d(x_{m_k+1}, x_{n_k+1})\}$ converge to δ as $k \to \infty$. Now,

$$\begin{aligned}&d(T^{m_k}x_0, S^{n_k}x_0) \leq d(T^{m_k}x_0, T^{n_k}x_0) + d(T^{n_k}x_0, S^{n_k}x_0) \text{ and}\\&d(T^{m_k}x_0, T^{n_k}x_0) \leq d(T^{m_k}x_0, S^{n_k}x_0) + d(S^{n_k}x_0, T^{n_k}x_0) \text{ for all } k \geq 1.\end{aligned} \tag{23}$$

Thus by taking $k \to \infty$ we have $\lim_{k\to\infty} d(T^{m_k}x_0, S^{n_k}x_0) = \delta = \lim_{k\to\infty} d(x_{m_k}, x_{n_k})$. Similarly we can show that $\lim_{k\to\infty} d(T^{m_k+1}x_0, S^{n_k+1}x_0) = \delta =$
$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1})$. Now,

$$\begin{aligned}&\zeta(d(T^{m_k+1}x_0, S^{n_k+1}x_0), d(T^{m_k}x_0, S^{n_k}x_0))+\\&K[d(T^{m_k}x_0, T^{m_k+1}x_0) + d(S^{n_k}x_0, S^{n_k+1}x_0)] \geq 0 \text{ for all } k \geq 1\\\Rightarrow &\limsup_{i\to\infty} \zeta(d(T^{m_k+1}x_0, S^{n_k+1}x_0), d(T^{m_k}x_0, S^{n_k}x_0)) \geq 0, \text{ a contradiction.}\end{aligned} \tag{24}$$

Therefore $\{x_n\}$ is Cauchy in X and due to the completeness of X there exists $u \in X$ such that $T^n x_0 \to u$ as $n \to \infty$. Since $\lim_{n\to\infty} d(T^n x_0, S^n x_0) = 0$ we have also $S^n x_0 \to u$ as $n \to \infty$. As T, S are either k-continuous or orbitally continuous in X it follows that $Tu = u = Su$ i.e. u is a common fixed point of T and S in X. Now we prove the uniqueness of the common fixed point of T and S. Let v be another common fixed point of T and S in X then we have

$$\zeta(d(Tu, Sv), d(u, v)) + K[d(u, Tu) + d(v, Sv)] \geq 0, \tag{25}$$

implying that $0 \leq \zeta(d(u, v), d(u, v)) < d(u, v) - d(u, v) = 0$, which can not possible. Hence T and S have a unique common fixed point in X. □

Corollary 2.1. *(a) If we consider $\zeta(t, s) = \lambda s - t$, $\lambda \in [0, 1)$ then the contractive condition* (2.1) *reduces to*

$$d(Tx, Ty) \leq \lambda d(x, y) + K[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X, \text{ where } K \geq 0. \tag{26}$$

(b) If we take $\zeta(t, s) = \varphi(s) - t$, $\varphi : [0, \infty) \to [0, \infty)$ is upper semi-continuous and $\varphi(s) < s$ then the contractive condition (2.1) *reduces to*

$$d(Tx, Ty) \leq \varphi(d(x, y)) + K[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X, \text{ where } K \geq 0. \tag{27}$$

(c) Let us put $\zeta(t, s) = s\varphi(s) - t$, where $\varphi : [0, \infty) \to [0, 1)$ is a mapping such that $\limsup_{t\to r^+} \varphi(t) < 1$, for all $r > 0$ then the contractive condition (2.1) *reduces to*

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + K[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X, \text{ where } K \geq 0. \tag{28}$$

Therefore Theorem 2.1 *generalizes Theorem* 2.6 *of* [9], *Theorem* 2.2 [8] *and Theorem* 2.1 [8] *which are proved by Górnicki.*

Corollary 2.2. *If we take* $\zeta(t,s) = \lambda s - t,\ \lambda \in [0,1)$ *then the contractive condition* (21) *reduces to*

$$d(Tx,Sy) \leq \lambda d(x,y) + K[d(x,Tx) + d(y,Sy)] \text{ for all } x,y \in X, \text{ where } K \geq 0. \tag{29}$$

Thus Theorem 2.2 *of* [6] *follows from our Theorem* 2.3.

Corollary 2.3. *If we take* $\zeta(t,s) = \lambda s - t,\ \lambda \in [0,1)$ *then the contractive condition* (16) *reduces to*

$$d(Tx,Ty) \leq \lambda d(Fx,Fy) + K[d(Fx,Tx) + d(Fy,Ty)] \text{ for all } x,y \in X, \quad \text{where } K \geq 0. \tag{30}$$

Hence Theorem 2.1 *of* [16] *follows from our Theorem* 2.2.

Example 2.2. (a) *Let* $X = \mathbb{R}_0^+(= [0,\infty))$ *with the usual metric and* $T : X \to X$ *be defined as in Example 2.1. Then* T *is a* $\mathcal{Z}$*-Proinov-Górnicki contractive mapping for* ζ_1 *given in Example 1.1 and* $K = 1$. *Here we see that* X *is complete and* T *satisfies all the conditions of Theorem 2.1.* 0 *is the unique fixed point of* T *in* X. *Also it is to be noted that* T *does not satisfy contractive condition (4).*

(b) *Let* $X = [0,4]$ *together with the usual metric and* $T : X \to X$ *be defined by*

$$T(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 2 \\ x-2 & \text{if } 2 < x \leq 4 \end{cases} \tag{31}$$

Then T *is a* $\mathcal{Z}$*-Proinov-Górnicki contractive mapping for* $\zeta(s,t) = s - \log(1+s) - t$ *for all* $s,t \geq 0$ *and* $K > 1$. *Moreover it satisfies all the additional conditions of Theorem 2.1. Here* X *is complete and* T *has a unique fixed point* 2.

Example 2.3. *Let* $X = \mathbb{R}_0^+(= [0,\infty))$ *together with the usual metric structure. Let us define* $T, F : X \to X$ *by* $T(x) = \frac{x}{x+1}$ *and* $F(x) = \frac{3x}{x+1}$ *for all* $x \in X$. *Then* T *and* F *satisfy the contractive condition (16). Here* T *and* F *fulfil all other conditions of Theorem 2.2 and it is seen that* 0 *is unique common fixed point of* T *and* F.

Example 2.4. *Let* $X = \{0\} \cup \{\frac{1}{n} : n \geq 1\}$ *endowed with the usual metric structure. Define* $T, S : X \to X$ *by* $T(0) = 0$, $T(\frac{1}{n}) = \frac{1}{n+1}$ *and* $S(0) = 0$, $S(\frac{1}{n}) = \frac{1}{n+2}$ *for all* $n \in \mathbb{N}$. *Then* T *and* S *satisfy the contractive condition (21) for* ζ_1 *given in Example 1.1 and for* $K = 2$. *Also* T *and* S *satisfy all other conditions of Theorem 2.3 and we see that* 0 *is unique common fixed point of* T *and* S. *Moreover it is seen that* T *and* S *do not satisfy contractive condition (29).*

Remark 2.1. *Theorem 2.1 gives us a totally new answer to the once open question of B. E. Rhoades* [13] *on the existence of contractive mappings which can be discontinuous at their fixed points.*

3. Conclusions

This paper deals with a new generalization of Kannan contractive mappings as well as Górnicki contractive mappings. Our work shows that, some times contractive conditions and completeness of the underlying spaces together can not ensure the existence of fixed points of mappings. There is a huge contribution of the asymptotic regularity and continuity of the considered mapping on the existence of fixed points. Also with the help of our mapping we can successfully extend the range of the constant used in Kannan contractive mapping. Finally Example 2.1 (b) increases the importance of our defined mapping, which provides discontinuity at its fixed point.

REFERENCES

[1] *Baillon, J.B., Bruck, R.E. and Reich, S.*, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, Houst. J. Math., **4** (1978), 1-9.

[2] *Browder, F. E. and Petryshyn, W. V.,* The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Am. Math. Soc., **72** (1966) 571-575.

[3] *R. K. Bisht,* A note on the fixed point theorem of Górnicki, JFPTA, **21**(54) (2019).

[4] *A. Chanda, L.K. Dey and S. Radenivić*, Simulation functions: a survey of recent results, RACSAM, **113** (2019) 2923-2957.

[5] *Jungck, G.*, Compatible mappings and common fixed points, Int. J. Math. Sci. **9**(4) (1986), 771-779.

[6] *A. R. Khan and D. M. Oyetunbi,* On some mappings with a unique common fixed point, JFPTA, **22**(47) (2020).

[7] *F. Khojasteh, S. Shukla and S. Radenović,* A new approach to the study of fixed point theory for simulation functions, Filomat, **29**(6) (2015), 1189-1194.

[8] *J. Górnicki,* On some mappings with a unique fixed point, JFPTA, **22**(8) (2020).

[9] *J. Górnicki,* Remarks on asymptotic regularity and fixed points, JFPTA, **21**(29) (2019).

[10] *Panja, S., Roy, K., Saha, M. and Bisht, R.K.,* Some fixed point theorems via asymptotic regularity, Filomat, 34:5 (2020), 1621-1627.

[11] *Pant, Abhijit and Pant, R. P.,* Orbital continuity and fixed points, Filomat 31:11 (2017), 3495-3499.

[12] *S. Radenović, F. Vetro and J. Vujaković,* An alternative and easy approach to fixed point results via simulation functions, Demonstr. Math., **50** (2017), 223-230.

[13] *B. E. Rhoades,* Contractive definitions and continuity, Contemp. Math., **72** (1988), 233-245.

[14] *K. Roy, S. Panja, M. Saha and R. K. Bisht*, On common and sequential fixed points via asymptotic regularity, CKMS, **37**(1) (2022), 163-176.

[15] *Shastri, K.P.R., Naidu, S.V.R., Rao, I.H.N. and Rao, K.P.R.,* Common fixed points for asymptotically regular mappings, Indian J. Pure Appl. Math., **15**(8) (1984), 849-854.

[16] *R. K. Bisht and N. K. Singh*, On asymptotic regularity and common fixed points, J. Anal., **28** (2020), 847-852.