APPLICATION OF FRACTIONAL DERIVATIVE TO THE RELAXATION OF LASER TARGET

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In această lucrare se aplică derivata fracţionară la modelarea procesului de relaxare a temperaturii unei ţinte laser. Pentru a descrie relaxarea temporală a unui fir în urma iradierii cu un puls laser se propune o ecuaţie diferenţială fracţionară. Prin rezolvarea acestei ecuaţii diferenţiale cu derivate fracţionare se obţin funcţia de transfer a temperaturii, răspunsul impuls şi soluţia generală. Relaxarea temporală a temperaturii ţintei este exprimată prin funcţii Mittag-Leffler care prezintă o relaxare polinomială în timp lung.

This paper focuses on the relation between fractional derivative and temperature relaxation of a wire after irradiation with a short laser pulse. We propose a fractional relaxation equation from which one obtains the transfer function of temperature, the impulse response. The general solution of fractional differential equation presents specific characteristics. In our model the temporal relaxation of target temperature is given by a linear combination of Mittag-Leffler functions which provides a simple generalization of the classical exponential function and describes the dynamic response of the temperature relaxation.

Keywords: Fractional Caputo derivative, Mittag-Leffler function, relaxation of temperature, laser target

1. Introduction

New technologies based on the interaction of ultra short laser pulses with the layer sequence of photonic crystals and quantum wells has allowed a huge further step to be made in temporal resolution at attoseconds level of physical and chemical processes [1]. The interaction of these pulses with matter is characterized by the proper time relaxation. A fundamental question of relaxation of energy of laser pulse in complex materials such as biotissue, photonic crystals and more general dielectrics, conductors and magnets is how to model the mechanical, thermal and electromagnetical relaxations of the target. For example, one of the applications is to produce under diffraction limit structures whose index of refraction is different from that of the sample. In this case it has been proposed

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that the laser induced strain field is responsible for the localized change of the material densification and heating [2]. Other example is the analyses of laser ablation of metals based on the so-called two –temperature model. It is based on the coupled temperature evolution of the electronic and atomic subsystems [3].

Unitl now, considerable amount of research in fractional calculus was published in physics literature [4], [5]. There in no doubt that fractional calculus has become a new mathematical method for solution of diverse problems in science and engineering. For example, almost all deformed materials exhibit both elastic and viscous properties through simultaneous storage and dissipation of mechanical energy. So any viscoelastic material may be treated based on a fractional equation of the dynamic connection between the stress and the strain. The analysis reveals that the fractional calculus models of viscoelastic behavior are much more satisfactory then the previously adopted classical models of viscoelasticity [6].

In this paper we discuss the application of fractional derivative to temporal relaxation of a laser target, and we show that fractional mathematical technique can be used for description of the temperature relaxation of a laser target. We propose a fractional relaxation equation from which one obtains the transfer function of temperature, the impulse response. The general solution of the fractional differential equation presents specific characteristics. In our model the temporal relaxation of the target is given by a linear combination of Mittag-Leffler functions. Note that the Mittag-Leffler function exhibits exponential behavior at short times and power-law relaxation (polynomial memory) at long times.

2. Fractional relaxation equation

There are many relaxation phenomena in nature whose relaxation function obeys the simple approximate equation

$$\tau \frac{d}{dt} f(t) + f(t) = 0 \tag{1}$$

with the exponential normalized solution

$$f(t) = \exp(-t/\tau) \tag{2}$$

and the relaxation time τ . For example, in dielectrics, the equation (1) is known as the Debye type relaxation equation. In this case the relaxation function is the dielectric displacement or polarization.

We define a fractional derivative $d^\alpha / dt^\alpha \equiv D^\alpha$ as an operator. The equality

$$D^\alpha E_\alpha(t^\alpha) = E_\alpha(t^\alpha) \tag{3}$$

defines $E_\alpha(t^\alpha)$ as its eigenfunction [7]. This function is known as Mittag- Leffler function and is represented by the polynomial series:

$$E_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{(\alpha k)!} \tag{4}$$

If $\alpha = 1$ in equation (4) the Taylor series of exp(t) is obtained. Therefore, Mittag-Leffler function can be thought as a generalized exponential function.

The composite fractional relaxation equation is obtained starting from the simple relaxation equation (1) in the form

$$\tau_i \frac{d}{dt} f(t) + (\tau_f)^\alpha \frac{d^\alpha}{dt^\alpha} f(t) + f(t) = Q(t) \tag{5}$$

with α a real number, $0 \le \alpha \le 1$ and Q(t) a given continuous function, with $t \ge 0$, which describes the source. τ_i is the coherent relaxation time, which yields an exponential relaxation and τ_f is an incoherent relaxation time, which acts for a polynomial memory of Mittag-Leffler type function. For $\alpha = 0$, equation (5) is reduced to a well-known one.

The fractional derivative from equation (5) is a Caputo fractional derivative of order α definite by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^1(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1 \tag{6}$$

where $f^1(\tau) \equiv D^1 f$ is the derivative of order 1. Wherever we use the operator $\frac{d^\alpha}{dt^\alpha} \equiv D^\alpha$ we tacitly assume the absolute integrability of the derivative of order 1. We remark the non-local character of Caputo fractional derivative, because it explicitly involves an integral which implies that the result depends not just on the values of f at the given t, but also on the whole stipulated range from 0 to t.

We easily recognize that in general the standard fractional derivative Riemann-Liouville of order α is defined by

$$D_R^\alpha = D^1 J^{1-\alpha} f(t), \text{where } J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, t > 0 \tag{7}$$

and it is different from Caputo fractional derivative: $D^\alpha f(t) = J^{1-\alpha} D^1 f(t)$.

The fractional differential equation (5) with Caputo derivative appears more suitable to be treated by Laplace transform technique because it requires the knowledge of the initial values of the function and its integer derivatives. The Laplace transform of Caputo derivative is given by

$$L(D^{\alpha} f(t)) = s^{\alpha} L(f(t)) - \sum_{k=0}^{m-1} f^{k}(0^{+}) s^{\alpha-1-k}, m-1 < \alpha < m, m = 0,1,2... \quad (8)$$

where $L(f(t)) = \int_0^{\infty} e^{-st} f(t)dt \equiv F(s)$ L is the Laplace transform.

The interaction of laser beam with a variety of targets shows a relaxation with memory. For example, laser cooling, the photorefractive effect, interferometry, optical diode, optical transistor and other laser interaction with solids and plasmas can be analyzed by fractional calculus.

3. Transfer function of target temperature

We first consider one long metallic wire with the diameter D which is irradiated with a laser pulse with the temporal shape described by function $\delta(t)$. The goal is to determine the dynamic (frequency) response of this laser target presented in figure 1. There are experimental difficulties regarding the time response of temperature measurement device, but with fast submicrometer thermocouples, thin film semiconducting thermistors, fluorescence thermometers, and Superconducting Transition Edge Sensors (TES) this problem can been overtaken.

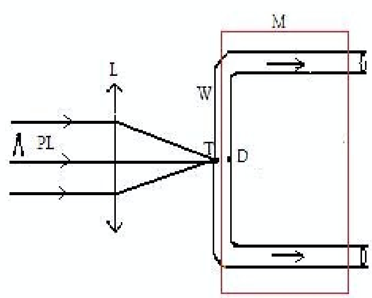


Fig. 1. PL-laser pulse, L-lens, T-source, W-wire, D-detector, M-medium

Temperature temporal relaxation by heat exchange between an ordinary wire and its surroundings can be written according to the classical theory in the following form:

$$T(t)/T_0 = \exp(-t/\tau) \quad (9)$$

where $\tau = D^2 c\rho / k_m$ is response time, c is the specific heat of the metal, ρ the density of the metal, T the difference between the medium and wire temperature, D the wire diameter, and k_m is thermal conductivity of surrounding medium. For example, a copper wire of diameter D=20 μm in water has a response time

$\tau = 2ms$. The response time of submicrometer thermocouple is as small as $\tau = 0.5\mu s$.

In this paper we assume that the conducted heat rate of one semi-infinite wire is given by

$$Q_c(t) = k/d^{\alpha} D_t^{\alpha} T(t) \tag{10}$$

were k is the thermal conductivity, d is the thermal diffusivity, $0 < \alpha \le 1$ D_t^{α} is the fractional derivative operator and T is temperature of the target in neighborhood of the laser focus. For $\alpha = 1$ fractional derivative operator is the ordinary derivative operator.

In this case the time domain behavior of wire is described by the following heat rate equation:

$$hA(T_s(t) - T(t)) = 2\, k/d^{\alpha} D_t^{\alpha} T(t) + \rho c D_t^1 T(t) \tag{11}$$

were h is the heat transfer coefficient from a source having a temperature $T_{s,}$ to the neighborhood through the surface area A of the laser-heated region and ρc is the product of the target mass and the specific heat of the material. To find out the transfer function, the effects of initialization are not required, therefore, all *T(0)* are zero. Equation (11) is in the shape of equation (5).

Taking the Laplace transform of this equation we obtain the transfer function as

$$H(s) \equiv \frac{T(s)}{T_s(s)} = \frac{1}{(\frac{\rho c}{hA})s + (\frac{2k}{hAd^{\alpha}})s^{\alpha} + 1} \tag{12}$$

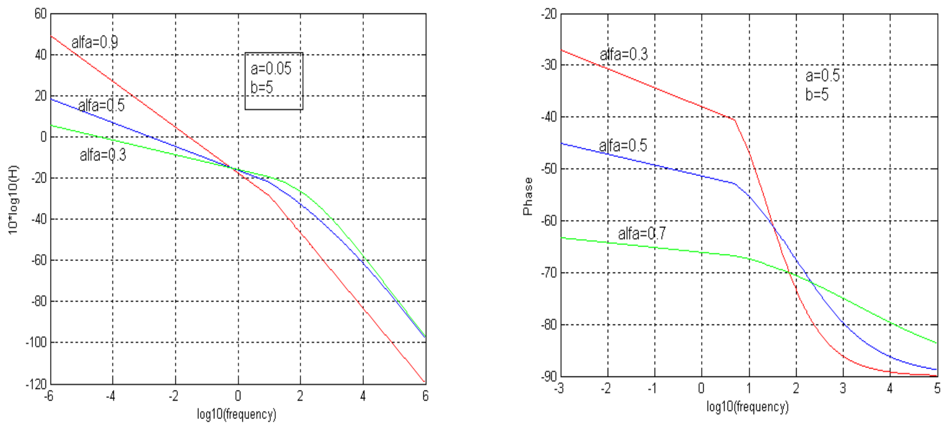
In the Fourier space, the magnitude $H(\omega)$ and phase angle $\varphi(\omega)$ are determined by letting $s = i\omega$ in equation (12), and noting that $i^{\alpha} = \cos\alpha\, \pi/2 + i \sin\alpha\, \pi/2$.

For example, if $\alpha = 1/2$, $a = \frac{\rho c}{hA} = 0.005$ and $b = \frac{2k}{hAd^{1/2}} = 5.0$ equation (12) yields

$$H(\omega) = \frac{1}{a^2\omega^2 + b^2\omega + 2ab\omega^{3/2}\sin\frac{\pi}{4} + 2b\omega^{1/2}\cos\frac{\pi}{4} + 1} \tag{13}$$

$$tg\varphi(\omega) = \frac{a\omega + b\omega^{1/2}\sin\frac{\pi}{4}}{b\omega^{1/2}\cos\frac{\pi}{4} + 1} \tag{14}$$

Fig. 2 shows the magnitude $H(\omega)$ and phase angle $\varphi(\omega)$ of the transfer function in the form of a Bode plot for $\alpha = 0.3, 0.5, 0.7$, $\frac{\rho c}{hA} = a = 0.05$ and $\frac{2k}{hAd^{1/2}} = b = 5.0$.



a b

Fig. 2:Amplitude and phase of transfer function: a – amplitude, b - phase

The response shows two distinct asymptotes; in the low frequency range a slope of -10db/decade corresponds to the $1/s^{1/2}$ behavior and a slope of -20db/decade for frequencies above 10^2 radians/second corresponds to 1/s behavior.

4. Impulse Response

With a transfer function as in equation (12) we can perform the inversion quite easily, by following the steps: 1) Transform *H(s)* into *H(z)*, by substitution of s^{α} with z. 2) Perform the expansion of *H(z)* in partial fractions. The denominator polynomial in *H(z)* is the indicial polynomial. We must use only the zeros of the indicial polynomials, that are really in the principal Riemann surface, $\{z : -\pi \le \arg(z) < \pi\}$, because only this one leads to a real system. 3) Substitute back s^{α} for z, to obtain the partial fractions in the form: $F(s) = 1/(s^{\alpha} - a)^k$, $k = 1,2,...$. 4) invert each partial fraction. 5) Add the different partial impulse responses.

In our case, we rewrite the equation (9) in the form:

$$H(z) = \frac{1}{az^2 + bz + 1} = \frac{A}{z - z_1} + \frac{B}{z - z_2} \tag{15}$$

$$H(s) = \frac{A}{s^\alpha - z_1} + \frac{B}{s^\alpha - z_2} \tag{16}$$

with z_1 and z_2 the roots of equation $az^2+bz+1=0$ and $-A=B=1/(z_1-z_2)$. By Inverse Laplace Transform (ILT) of a partial fraction as $H_1(s) = 1/(s^\alpha - z_1)$ the impulse response $h_1(t)$ is a linear combination of $q=1/\alpha$, Mittag-Leffler function:

$$h_1(t) = \frac{1}{z_1^j} \sum_{j=1}^{q} z_1^j E_{1-j\alpha}(t, z_1^q) \tag{17}$$

with α rational, z_1, z_2 real numbers, and Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha z + 1)} \ , \ \alpha > 0, z \in C \tag{18}$$

For example, if $\alpha = 1/2$ the Impulse function is given by [11]

$$h(t) = C_1 E_{1/2}(z_1\sqrt{t}) + C_2 E_{1/2}(z_2\sqrt{t}) \tag{19}$$

where $C_1 = z_1/(z_1 - z_2)$ and $C_2 = -z_2/(z_1 - z_2)$.

In the figure 3 is shown the function

$$E_{1/2}(-t^{1/2}) = \exp(-t^{1/2})\left[1 - erf(\pm t^{1/2})\right] = \exp(-t^{1/2}) erfc(-t^{1/2}) \tag{20}$$

where *erf (erfc)* denote the error (complementary) function defined as

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du \ , erfc = 1 - erf(z) .$$

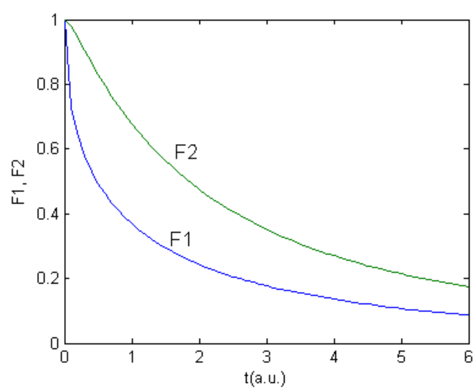


Fig. 3. Exponential function (F1) and Mittag-Leffler function (F2)

5. Solutions of composite fractional relaxation equation

Equations (5) and (11) can be thought as a generalization of the ordinary differential equation related to relaxation phenomena. We note that when $\alpha = 1$ these equations reduce to an ordinary differential equation whose solution can be expressed in terms of a solution of the homogeneous equation and of one particular solution of the inhomogeneous equation. We rewrite the equation (5) in the dimensionless form:

$$\frac{du(\tau)}{d\tau} + a\frac{d^{\alpha}u(\tau)}{d\tau^{\alpha}} + u(\tau) = q(\tau) \tag{21}$$

where $t = \tau_i \tau$, $u(\tau) = f(\tau_i \tau) = f(t)$, $a = (\frac{\tau_f}{\tau_i})^{\alpha}$ and $q(\tau) = Q(t)$.

We shall apply the method of the Laplace transform (LT) to solve the fractional differential equation (21). Using the rule

$$L(\frac{d^{\alpha}u(\tau)}{d\tau^{\alpha}}) = s^{\alpha}L(u(\tau)) - \sum_{k=0}^{m-1} u^{k}(0^{+})s^{\alpha-1-k}, m-1 < \alpha \le m$$

we are led to the algebraic equation

$$U(s) = \frac{c_0(1 + as^{\alpha-1})}{s + as^{\alpha} + 1} + \frac{Q(s)}{s + as^{\alpha} + 1} \tag{22}$$

where *U(s)*, *Q(s)* are the LT of $u(\tau)$, respectively $q(\tau)$ and c_0 is the initial value of $u(\tau)$, namely $c_0 = u(0^{+})$. Putting

$$U_0(s) = \frac{c_0(1 + as^{\alpha-1})}{P(s)}, U_{\delta}(s) = \frac{1}{P(s)} \tag{23}$$

where $P(s) = s + as^{\alpha} + 1$ we find, from ILT, the general solution of equation (21)

$$u(\tau) = c_0 u_0(\tau) + \int_0^{\tau} q(\tau - \tau')u_{\delta}(\tau')d\tau' \tag{24}$$

In equation (24), $u_0(\tau)$ is the fundamental solution, and $u_{\delta}(\tau)$ is the impulse response solution.

The problem to obtain $u_0(\tau)$ as the ILT of $U_0(s)$ is solved in [8] and given by

$$u_0(\tau) = \int_0^{\infty} e^{-r\tau} H^{1}_{\alpha,0}(r, a)dr \tag{25}$$

with

$$H^1_{\alpha,0}(r,a) = \frac{1}{\pi}\frac{ar^{\alpha-1}\sin(\alpha\pi)}{(1-r)^2 + a^2r^{2\alpha} + 2(1-r)ar^{\alpha}\cos(\alpha\pi)}, \tag{26}$$

the spectral function of $u_0(\tau)$. The figure 4 shows the spectral function $H^1_{\alpha,0}(r,a)$ for some values of α and a. One observes that the spectral function is positive for any $r>0$. This is a sufficient condition for the function $u_0(\tau)$ to be completely monotone for $r>0$. If a=1.5 and $\alpha > 0.51$, the spectral function $H(r)$ shows resonant behavior with a maximum at 0.3<r<0.4. This resonance propagates to the fundamental solution $u_0(\tau)$. The resonant behavior is not present for a=1.5 and $\alpha < 0.5$, the fundamental solution having an essential feature of the $\exp(-\lambda\tau)$ type. If the parameter $a \in (0.5,3)$, the spectral function $H^1_{\alpha,0}(r,a)$ looks like that corresponding to the case of a=1.5.

The determination of $u_\delta(\tau)$ is straightforward by derivative of $u_0(\tau)$, namely $u_\delta(\tau) = -u_0(\tau)$.

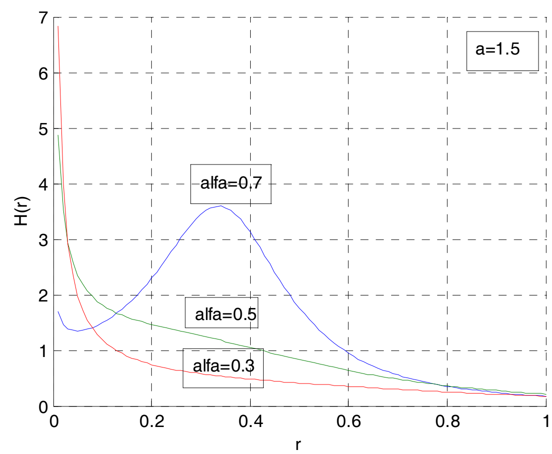


Fig. 4. Spectral function $H^1_{\alpha,0}(r,a)$

If α is rational number, namely $\alpha = p/q$ where $p,\ q \in N$ are assumed to be relative prime, a factorization of $P(s)$ is possible. In these cases the solution can be expressed in terms of a linear combination of q, Mittag-Leffler functions of fractional order $1/q$ as in paragraph 4.

Several forms of fractional differential equation have been proposed as models in physics, and there has been significant interest in developing numerical schemes for their solution [13], [14].

6. Conclusions

We show that fractional mathematical technique can be used for description of the temperature relaxation of a laser target. We propose a fractional relaxation equation from which one obtains the transfer function of temperature and the impulse response. The general solution of this fractional differential equation is given by a linear combination of Mittag-Leffler functions. Mittag-Leffler function provides a simple generalization of the classical exponential function and describes the dynamic response of the laser target temperature relaxation. We note that a possible physical interpretation of the fractional relaxation equation can be connected with memory effects or fractal properties of the medium [11].

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