THE APPLICATION OF GROUP THEORY IN THE GLOBAL ANALYSIS OF MECHANISM KINEMAT CHARACTERISTICS

Dengfeng ZHAO[1], Guoying ZENG[2], Yubin LU[3],Guolu MA[4]

The global analysis of mechanism kinematic characteristics is commonly to obtain the distribution characteristic of global mechanism kinematic characteristics, based on the adjacency relation of numerous simple geometries, which are divided from the mechanism parameter space by using the singularity condition of mechanism motion. In this analysis process, the calculation workload will be rapidly increased with increasing the number of mechanism parameters. By using group theory for the symmetry analysis of mechanism parameters, the calculation workload can be greatly reduced. In this paper, firstly the transformation group and its basic properties are introduced. Then the transformation group of mechanisms is defined, and the transformation group of planar single-loop mechanisms and sphere single-loop mechanisms is analyzed. Finally, the application of mechanism transformation group in mechanism global characteristics analysis is discussed. The results show that the mechanism transformation group is a necessary mathematical tool for the mechanism global characteristics analysis, otherwise the whole distribution characteristics of global mechanism performance could not be grasped.

Keywords: transformation group; symmetry; planar mechanisms; spherical mechanisms

1. Introduction

Traditional mechanisms analysis is to solve variation with time about motion parameters or to solve performance parameters (such as pressure angle, motion decoupling, etc.) of the different motion location of the mechanism, under the condition that links connection relationship, links size and drive condition have been determined. Whereas, global analysis of mechanisms performance is to discuss distribution characteristic and the whole evolution process with links size for the whole structure and motion parameters of mechanism in the whole range of the parameters of links size, under the condition that links connection relationships have

[1] Prof., Southwest university of science and technology, Sichuan, PR China, e-mail: 1264879506@qq.com
[2] Prof., Southwest university of science and technology, Sichuan, PR China, e-mail: zgywf@163.com
[3] Prof., Southwest university of science and technology, Sichuan, PR China, e-mail: 289232944 @qq.com
[4] Doc., Southwest university of science and technology, Sichuan, PR China, e-mail: 151251069 @qq.com

been only determined. Traditional mechanisms analysis is engineering science problems based on the kinematics and dynamics, whose theoretical basis is complete and the analysis method is also quite mature. Even considering links deformation, friction, collision contact and so on, many software applications can solve very complicated problems of mechanisms analysis. Global analysis of mechanism kinematic characteristics is based on modern mathematics theories such as topology, singular bifurcation theory and group theory and so on.

The global kinematic characteristics of mechanisms were closely related with the singularity of mechanisms. However, the current researches only focus on singularity of mechanisms. Adequate study has been conducted on almost all types of mechanisms and analysis methods. In the early time, Ting et al. [1] studied the rotatability law for N-bar kinematic chains by a traditional method. Afterwards the researches on the singularity of mechanisms are widely conducted. Alici [2] focused on the determination of singularity contours for a manipulator. Jiang et al. [3] gave the singularity orientation of the Gough–Stewart platform. Hang et al. [4] presented the decoupling conditions of spherical parallel mechanisms. Wolf et al. [5] analyzed the singularities of a three degree of freedom spatial Cassino Parallel manipulators. Yangmin Li et al. [6] performed dynamics analysis of a three-prismatic-revolute-cylindrical (3-PRC) parallel kinematic machine (PKM).

Researches on the relations between the global motion characteristics and the singularity of mechanisms are few. Gao et al. [7] introduce the concept of solution space to the mechanisms evolution analysis, the workspace atlas of parallel planar manipulators were obtained. Zhao et al. [8,9,10,11,12] using the convex hull division method explored the classification method of single-loop planar and spherical mechanisms.

The global analysis of mechanism kinematic characteristics should include the following three main parts. One is to analysis the whole motion range of mechanism parameters. The second is the global distribution characteristics of mechanism kinematic characteristics in the whole motion range. The third is the whole evolution process of the motion range and distribution characteristics with the structure and motion parameters of mechanisms. The basic methods of mechanism global analysis is dividing the structural parameter space to obtain mechanism dimensional type with different performance characteristics, or dividing the motion parameter space to obtain the range of motion parameters with significantly different performance distribution characteristics. Then the global mechanism performance is grasped, according to the mechanism performance characteristics in the above range and the relationship among this range. In the process of division, many division conditions are involved, and multitudinous geometric objects adjacent to each other are generated. On the other hand, all kinds of symmetry always exist in mechanisms, which are important to clarify the relationship among geometries. Introducing the group theory, as a powerful mathematical tool for analyzing complex symmetry, is necessary to the global performance analysis.

This paper includes four parts. The first is transformation group and its basic properties, where group theory terms and their meaning involved in this study are introduced. The second is defining the transformation group of mechanisms, and the transformation group of planar single loop and spherical single loop mechanisms are presented. Then possible applications of transformation group for the analysis of mechanisms are given. Finally, results show that the mechanism transformation group is a necessary tool for the global analysis of mechanism kinematic characteristics. Otherwise, it will be impossible to grasp the overall distribution of global performance.

2. Transformation group and its basic properties

Let X be a thing, $g(X)$ is an operation on X, satisfying $X=g(X)$, where $g(X)$ is called the symmetric transform of X. If both g_i and g_j are symmetric transforms of X, the "multiplication" $g_i g_j$ is defined as the composite transformation $g_i(g_j(X))$, usually $g_i g_j \neq g_j g_i$. The set of all symmetric transforms of X, $\mathbb{G}(X)=\{e, g_2, \ldots, g_i, \ldots\}$, is called as the transformation group of X. [13]

Transformation group $\mathbb{G}(X)$ satisfies the following four group axioms of abstract group. 1) Closure, $g_i g_j \in \mathbb{G}$, $g_i, g_j \in \mathbb{G}$; 2) Associativity, $g_i(g_j g_k) = (g_i g_j)g_k$, $g_i, g_j, g_k \in \mathbb{G}$; 3) The existence of an identity element (i.e., identity transformation), e: $g_i e = e g_i = g_i$, $g_i \in \mathbb{G}$; 4) Inverse, for each $g_i \in \mathbb{G}$, there exists $g_i^{-1} \in \mathbb{G}$, thus $g_i g_i^{-1} = g_i^{-1} g_i = e$. The number of elements in $\mathbb{G}$ is called the order of $\mathbb{G}$, written as $|\mathbb{G}|$. If $|\mathbb{G}|$ is finite, $\mathbb{G}$ is said to be a finite group. Otherwise, it is known as infinite group.

If a subset $\mathbb{H}$ of group $\mathbb{G}$ also forms a group, $\mathbb{H}$ is called a subgroup of $\mathbb{G}$, denoted as $\mathbb{H} \leq \mathbb{G}$. Obviously, e is the smallest subgroup, while group $\mathbb{G}$ is the biggest subgroup.

If $\mathbb{H} = \{e, h_2, \ldots, h_m\}$ is a subgroup of $\mathbb{G}$, for one element $x \in \mathbb{G}$, group $x\mathbb{H} = \{ x, xh_2, \ldots, xh_m \}$ is called left coset of $\mathbb{H}$. Left coset has the following basic properties,

(1)All left cosets of $\mathbb{H}$ have the same number of elements.

(2)If $g \in x\mathbb{H}$, there is $g\mathbb{H} = x\mathbb{H}$, namely all elements of cosets can be obtained by multiplying itself and $\mathbb{H}$. Any two of the left cosets $x\mathbb{H}$ and $y\mathbb{H}$, is precisely equal, or totally different.

(3)Mother group $\mathbb{G}$ could be divided into the set of left coset of subgroup $\mathbb{H}$. Namely, the order $|\mathbb{G}|$ must be the integer k times of the order $|\mathbb{H}|$, called k as the index *of* $\mathbb{H}$ in $\mathbb{G}$, namely the number of cosets of $\mathbb{H}$ in $\mathbb{G}$.

Subgroup $\mathbb{H}$ also has right coset $\mathbb{H}x$, which has the same properties with left coset, but may be not equal. If $x\mathbb{H} = \mathbb{H}x$, the $\mathbb{H}$ is called normal subgroup, then all cosets of $\mathbb{H}$ form a group which is called quotient group, written as $\mathbb{G}/\mathbb{H}$. Element of quotient group is each coset itself.

The orbital theory of transformation group based on subgroups and their cosets is the foundation of symmetry analysis.

Let A be a set, its transformation group is $\mathbb{G}(A)$, given an element $x \in A$, $\mathbb{H}(x)$ is the transformation group of x in $\mathbb{G}(A)$, namely satisfying $\mathbb{H}(x) = \{g \mid g \in \mathbb{G},\ g(x) = x\}$. The set $\{g(x) \mid g \in \mathbb{G}\}$ is defined as the group orbital of element x.

The group orbital of element x is given by the left coset of $\mathbb{H}(x)$ in $\mathbb{G}(A)$. The transformation group of $g(x)$ is the conjugate of $\mathbb{H}(x)$, $g\mathbb{H}g^{-1}$. If $\mathbb{G}$ is the finite group, the number *of* elements in the group orbital is $k = |\mathbb{G}|/|\mathbb{H}|$. Examples of typical groups are as follows,

(1) The set of all real numbers (or integers) with the addition and 0 as identity is a group. It is denoted by $\mathbb{R}_A$ (or $\mathbb{Z}_A$) and called the additive group of real number (or integer). Obviously, $\mathbb{Z}_A$ is a subgroup in $\mathbb{R}_A$

(2) The set of nonzero (or >0) real numbers with the multiplication and 1 as identity is a group. It is denoted by $\mathbb{R}_M$ (or $\mathbb{R}^+_M$) and called the multiplicative group of real number. Obviously, $\mathbb{R}^+_A$ is a subgroup in $\mathbb{R}^+_M$. the set {1, -1} is also a multiplicative group and is subgroup in $\mathbb{R}_M$.

(3) The set of $N \times N$ real orthogonal matrices is a group for the matrix product and identity the matrix $\boldsymbol{I}_n$. It is denoted by $\mathbb{O}_\mathrm{N}$ and called the orthogonal matrices group. The subset with determinant of 1 is a subgroup of $\mathbb{O}_\mathrm{N}$, denoted by $\mathbb{SO}_\mathrm{N}$ and called the special orthogonal matrices group.

(4) The operation of arranging elements of a set of N elements to a certain sequence is called permutation. All permutation of N-elements form a symmetry group $\mathbb{S}_N$, whose order is $|\mathbb{S}_N| = N!$, which contains abundant subgroups. For example, the transformation group of regular triangles is symmetry group $\mathbb{S}_3$ of three vertexes, which includes three rotation elements around the centroid 0°、120° and 240°, and three mirror transformations around the medians.

3. Transformation group of mechanisms

The mechanism parameter can be divided into motion space parameter $\boldsymbol{u} = (u_1, \ldots, u_U)$ changing with time and structural space parameter $\boldsymbol{v} = (v_1, \ldots, v_V)$ remaining constant during motion. The mechanism can be regarded as a constraint equation defined in motion space and structural space.

$$\boldsymbol{f}(u,v)^T = \begin{bmatrix} f_1(u_1,\cdots,u_U;v_1,\cdots,v_V) \\ \vdots \\ f_F(u_1,\cdots,u_U;v_1,\cdots,v_V) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{1}$$

The solution space of mechanisms is defined as the solution set of constraint equations in motion space, whose dimension is $D=U-F$. All characteristics of mechanism performance depend on the characteristics of the constraint equations, and completely reflect in the solution space. The singularity properties depend on the first-order deferential properties to motion parameters.

$$d\boldsymbol{f} = \boldsymbol{J}d\boldsymbol{u} \tag{2}$$

where $\boldsymbol{J}$ is the Jacobi matrix of $\boldsymbol{f}$ to $\boldsymbol{u}$, whose element $J_{ij}=\partial f_i/\partial u_j$. When $\boldsymbol{J}$ is an owe-rank matrix, the topology of the solution space may mutate, and the whole distribution characteristics of mechanism performances may mutate. At this time, the determinants of $F\times F$ square matrices taking F columns from J are all zero. The number of such conditions is $U!/F!/(U-F)!$, but the number of independent conditions is only $U-F+1$. Combining these conditions with the constraint equations eliminates motion parameters, and forms the singularity division condition of mechanism structural space,

$$\boldsymbol{F}^v(\boldsymbol{v}) = \begin{bmatrix} f_1^v(v_1,\cdots,v_V) \\ \vdots \\ f_{F^v}^v(v_1,\cdots,v_V) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$

Usually there are numerous division conditions of singularity which divide the structural space into many geometric objects corresponding to solution space with different topologic. These division conditions are the intrinsic condition of mechanism dimension classification.

For mechanisms with fixed structural parameters, the solution space and global structure of mechanism performance are determined. To find out the overall structure of the solution space, a straightforward method is dividing the solution space into geometric objects with inner performance being monotonous distribution, while the adjacency relationship of geometric objects reflects the global structure of solution space. The determinant of $F\times F$ square matrices taking F columns from J is zero, corresponding to the singularity condition of mechanism motion parameters. The number of such conditions is $U!/F!/(U-F)!$, which forms the division condition of solution space

$$\boldsymbol{F}^u(\boldsymbol{u}) = \begin{bmatrix} f_1^u(u_1,\cdots,u_U) \\ \vdots \\ f_{F^u}^u(u_1,\cdots,u_U) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$

For structural space and solution space, division conditions and division results are complex, and there are various kinds of symmetry. So the introduction of group theory for analyzing symmetry to analyze division conditions and division results is necessary.

For a nonsingularity transformation $(\boldsymbol{u}', \boldsymbol{v}')=\boldsymbol{T}_p(\boldsymbol{u}, \boldsymbol{v})$ of mechanism parameter, if meanwhile there is nonsingularity transformation $\boldsymbol{f}'=\boldsymbol{T}_f(\boldsymbol{f})$, $\boldsymbol{f}'^v=\boldsymbol{T}_f^v\left(\boldsymbol{f}^v\right)$ and $\boldsymbol{f}'^u=\boldsymbol{T}_f^u\left(\boldsymbol{f}^u\right)$ respectively satisfying the invariant condition in following composite transformations,

$$\boldsymbol{T}_f\boldsymbol{F}\boldsymbol{T}_p=\boldsymbol{F} \qquad \boldsymbol{T}_f^v\boldsymbol{F}^v\boldsymbol{T}_p=\boldsymbol{F}^v \qquad \boldsymbol{T}_f^u\boldsymbol{F}^u\boldsymbol{T}_p=\boldsymbol{F}^u \tag{5}$$

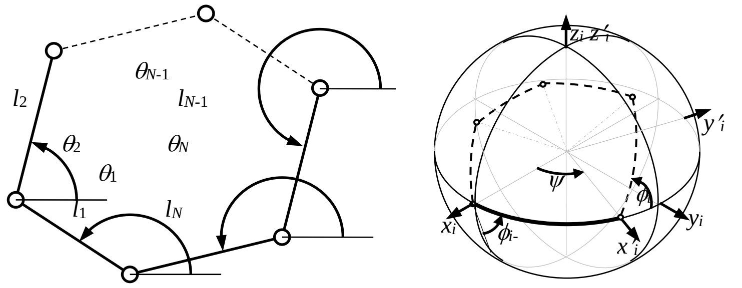
Then transformation $\boldsymbol{T}_p$ is called the symmetry transformation of mechanisms, transformation $\boldsymbol{T}_f$, $\boldsymbol{T}_f^v$ and $\boldsymbol{T}_f^u$ is called the dual transformation for constraint equations, structural space and solution space, respectively. All symmetry transformations of $\boldsymbol{T_P}$ form transformation group $\mathbb{G}(\boldsymbol{u},\boldsymbol{v})$ of mechanisms, all dual transformations of $\boldsymbol{T_F}, \boldsymbol{T_F^u}$ and $\boldsymbol{T_F^v}$ form dual the transformation group $\mathbb{G}(\boldsymbol{f})$, $\mathbb{G}(\boldsymbol{f}^v)$ and $\mathbb{G}(\boldsymbol{f}^u)$ of constraint equations, structural space and solution space, respectively. Transformation group and dual transformation group keep homomorphism.

The symmetry transformation of *Jacobi* matrix can be obtained by taking the derivative of symmetry transformation of constraint equations to motion parameter,

$$\left[\frac{\partial \boldsymbol{T_F}}{\partial \boldsymbol{F}}\right]\left[\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}'}\right]\left[\frac{\partial \boldsymbol{T_P}}{\partial \boldsymbol{u}}\right]=\left[\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}\right] \quad or \quad \left[\frac{\partial \boldsymbol{T_F}}{\partial \boldsymbol{F}}\right]\boldsymbol{J}'\left[\frac{\partial \boldsymbol{T_P}}{\partial \boldsymbol{u}}\right]=\boldsymbol{J} \tag{6}$$

Eq. (6) shows that the symmetry transformation and dual transformation of *Jacobi* matrix are the derivative of transformation of constraint equations, namely the *Jacobi* matrix inherit all symmetry features of constraint equations. In addition, because all division conditions include all singularity conditions of *Jacobi* matrix, the set of all division conditions naturally inherit the symmetry features of constraint equations.

N-links planar and spherical single-loop mechanisms shown in Fig.1 are taken as two examples to illustrate transformation group and dual transformation group of mechanisms.



(a) (b)

Fig.2 planar and spherical single-loop mechanism and its parameters

For a planar single-loop mechanism shown in Fig.1(a), the motion parameter is the rotation angle of links, $\boldsymbol{u}=(\theta_1,\ldots,\theta_N)$, the structural parameter is length of links, $\boldsymbol{v}=(l_1,\ldots,l_N)$, and the constraint equation is a closed condition,

$$\boldsymbol{f}(\boldsymbol{u},\boldsymbol{v})=\begin{bmatrix} l_1\cos\theta_1+\cdots+l_N\cos\theta_N \\ l_1\sin\theta_1+\cdots+l_N\sin\theta_N \end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{7}$$

The dimension of solution space (namely the degree of freedom) is *N*-2. For the spherical single-loop mechanism shown in Fig.1(b), the motion parameter is the relative rotation angle of each motion pair, $\boldsymbol{u}=(\phi_1,\ldots,\phi_N)$, the structural parameter is the angle between two motion pairs on each links, $\boldsymbol{v}=(\psi_1,\ldots,\psi_N)$. To analyze conveniently, coordinate systems of (O,x_i,y_i,z_i) and (O,x'_i,y'_i,z'_i) are established on each link. The constraint equation of this is a closed condition,

$$\prod_{i=0}^{N-1}\boldsymbol{\Phi}_{N-i}\boldsymbol{\Psi}_{N-i}=\boldsymbol{\Phi}_N\boldsymbol{\Psi}_N\cdots\boldsymbol{\Phi}_1\boldsymbol{\Psi}_1=I \tag{8}$$

where $\boldsymbol{\Psi}_i$ is the coordinate transformation matrix rotating ψ_i around z_i axis, and $\boldsymbol{\Phi}_i$ is the coordinate transformation matrix rotating ϕ_i. around x_i axis. Although Eq.(8) is a 3×3 matrix equation, the number of independent equations is 3 rather than 9, since all coordinate transformation matrixes are rotation matrix. The dimension of solution space (namely the degree of freedom) of this mechanism is N-3.

The *Jacobi* matrix of the planar single-loop mechanism can be easily obtained,

$$\boldsymbol{J}=\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}=\begin{bmatrix}-l_1\sin\theta_1 & \cdots & -l_N\sin\theta_N\\ l_1\cos\theta_1 & \cdots & l_N\cos\theta_N\end{bmatrix} \tag{9}$$

The determinant of a 2×2 matrix taking any two columns i, j from this *Jacobi* matrix is zero, singular division conditions with the number of $N(N$-1)/2 can be obtained,

$$l_i l_j\sin(\theta_i-\theta_j)=0 \qquad i,j\in\{1,2,\cdots,N\} \tag{10}$$

Eq.(10) requires that links i, j are positive or reverse parallel. By using parallel conditions of all links to eliminate motion parameters, the division conditions of structural space can be obtained,

$$\pm l_1\pm\cdots\pm l_N=0 \tag{11}$$

After the tedious mathematical derivation, the *Jacobi* matrix of spherical single loop mechanisms is obtained,

$$\boldsymbol{J}=\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}=\begin{bmatrix}x_{1,1} & \cdots & x_{N-1,1} & 1\\ y_{1,1} & \cdots & y_{N-1,1} & 0\\ z_{1,1} & \cdots & z_{N-1,1} & 0\end{bmatrix} \tag{12}$$

where $(x_{i,1},y_{i,1},z_{i,1})$ is the coordinate of motion pair i in the coordinate (O,x_1,y_1,z_1). The determinant of a 3×3 matrix formed by taking any three columns i,j,k from a *Jacobi* matrix of spherical mechanism is zero, singular division conditions with the number of $N(N$-1)(N-2)/6 can be obtained,

$$\begin{vmatrix}x_{i,1} & x_{j,1} & x_{k,1}\\ y_{i,1} & y_{j,1} & y_{k,1}\\ z_{i,1} & z_{j,1} & z_{k,1}\end{vmatrix}=0 \qquad i,j,k\in\{1,2,\cdots,N\} \tag{13}$$

Eq. (13) requires that three motion pair vectors i,j,k are coplanar. By using the coplanar conditions of all motion pairs to eliminate motion parameters, the division conditions of structural space can be obtained,

$$\pm\psi_1\pm\cdots\pm\psi_N=2k\pi \qquad k\in\{0,\pm1,\pm2,\cdots\} \tag{14}$$

The transformation group of the mechanism constraint equations depends on mechanism parameters and the form of constraint equations. But still there are some universal conditions as follows,

(1) Whole movement symmetry: mechanism performance has nothing to do with the position and direction of overall mechanism in three-dimensional physical space, so all transformations of mechanism translation, rotation and mirror in physical space are symmetry transformation, whose transformation group is motion group.

Motion parameters of planar single-loop mechanisms only have absolute rotation angle, so the whole translational invariance is certainly satisfied. Whole rotation transformation is that all motion parameters $(\theta_1,\ldots,\theta_N)$ meanwhile adding arbitrary angle θ_0 transform to be $(\theta_1+\theta_0,\ldots,\ \theta_N+\theta_0)$, whose transformation group is real additive group, written as $\mathbb{R}^+(\boldsymbol{u})$. Dual transformation is left multiplication rotation matrix, whose group is written as $\mathbb{O}_2(\boldsymbol{f})$. The invariance verification of the constraint equations is,

$$\begin{bmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} l_1\cos(\theta_1+\theta_0)+\cdots+l_N\cos(\theta_N+\theta_0) \\ l_1\sin(\theta_1+\theta_0)+\cdots+l_N\sin(\theta_N+\theta_0) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad (15)$$

For Eq.(10) and (11) of division conditions of structural space and solution space, the invariance is obvious, whose all dual transformations are identical transformation. The mirror transformation is that all motion parameters $(\theta_1,\ldots,\theta_N)$ is meanwhile transformed to $(-\theta_1,\ldots,-\theta_N)$, whose group is written as $\mathbb{I}(\boldsymbol{u})$. The dual transformation of constraint equations is f_1=-f_1, f_2=-f_2, whose group is written as $\mathbb{I}(\boldsymbol{f})$. The dual transformation of division conditions of solution space is $(-f^u_1,\ldots,-f^u_N)$, whose group is written as $\mathbb{I}(\boldsymbol{f^u})$.

Motion parameters of spherical single-loop mechanisms only have relative rotation angle of links, so the whole translational invariance and rotational invariance can be certainly satisfied. The whole mirror transformation is that all motion parameters $(\phi_1,\ldots,\phi_N)$ is meanwhile transformed to be $(-\phi_1,\ldots,-\phi_N)$, whose group is written as $\mathbb{I}(\boldsymbol{u})$. The invariance verification of the constraint equations is,

$$I^{\psi}\boldsymbol{\Phi}_N^{-1}I^{\psi}I^{\psi}\boldsymbol{\Psi}_N I^{\psi}\cdots I^{\psi}\boldsymbol{\Phi}_1^{-1}I^{\psi}\boldsymbol{\Psi}_1 I^{\psi} = \boldsymbol{\Phi}_N\boldsymbol{\Psi}_N\cdots\boldsymbol{\Phi}_1\boldsymbol{\Psi}_1 = I^{\psi}II^{\psi} = I \quad (16)$$

where I^{ψ} is special 3×3 diagonal matrix, whose element I^{ψ}_{11}= I^{ψ}_{22}=1, I^{ψ}_{33}=-1. Similarly, when all motion parameters $(\psi_1,\ldots,\psi_N)$ are simultaneously transformed to be $(-\psi_1,\ldots,-\psi_N)$, it becomes symmetry transformation, whose group is written as $\mathbb{I}(\boldsymbol{v})$. The dual transformation and invariance verification of the constraint equations are similar with Eq.(16), only replacing I^{ψ} with diagonal matrix I^{ϕ}, whose element I^{ϕ}_{11}=-1, I^{ϕ}_{22}=I^{ϕ}_{33}=1. The dual transformation and its invariance verification of Eq.(13) of solution space division conditions is similar with Eq.(16), thus omitted here. The invariance of solution space division conditions, Eq.(14), is obvious, whose dual transformation is identical transformation.

(2)Projective symmetry: Projective transformation is zooming all linear sizes of a mechanism with special size at same ratio, but the same ratio doesn't be required for the mechanism with different sizes. The transformation group is real multiplicative group. Projective transformation of planar single-loop mechanisms is that all structural parameters ($l_1,\ldots,l_N$) are multiplied by any non-zero real number λ. The dual transformation of constraint equations Eq.(7) and division condition Eq.(14) of structural space are all equation multiplied with 1/λ. The division condition of solution space has nothing to do with structural parameters, whose dual transformation is identical transformation. All parameters of spherical mechanisms are angle parameters, which certainly satisfy projective invariance.

⑶Inversion symmetry: If a parameter set ($p_1,\ldots,p_P$) replaced by ($-p_1,\ldots,-p_P$) is the symmetry transformation of mechanisms, which has inversion symmetry. The inversion transformation of planar mechanisms is that any parameter l_i is replaced by $-l_i$, and corresponding rotation angle θ_i is replaced by $\theta_i+\pi$, whose transformation group is written as $\mathbb{I}(l_i,\theta_i)$. The dual transformation of constraint equations is identical transformation. The dual transformation of structural space division conditions, Eq.(10), is the exchange of division conditions, whose l_i sign is opposite and other signs are the same. The dual transformation of solution space division conditions, Eq.(11), is division conditions related with θ_I multiplied with -1.

In spherical mechanisms, the inversion transformation of parameter ψ_i is ψ_i replaced by $-\psi_I$, and the rotation angle θ_i and θ_{i-1} replaced by $\theta_i+\pi$ and $\theta_{i-1}+\pi$, whose transformation group is written as $\mathbb{I}(\psi_i,\theta_i,\theta_{i-1})$. The dual transformation and the invariance verification of constraint equations are,

$$\boldsymbol{\Phi}_i(\phi_{i-1}+\pi)I^{\phi}I^{\phi}\boldsymbol{\Psi}_i^{-1}I^{\phi}I^{\phi}\boldsymbol{\Phi}_{i-1}(\phi_{i-1}+\pi)=\boldsymbol{\Phi}_i\boldsymbol{\Psi}_i\boldsymbol{\Phi}_{i-1} \tag{17}$$

Correspondingly, the dual transformation and the invariance verification of structural space division conditions is equal with Eq.(17). The dual transformation of structural space division conditions is the exchange of division conditions, whose ψ_i sign is opposite and other signs keep the same.

In spherical mechanisms, the inversion transformation of parameter ϕ_i is ϕ_i replaced by $-\phi_i$, and the rotation angle ψ_i and ψ_{i-1} replaced by $\psi_i+\pi$ and $\psi_{i-1}+\pi$, whose transformation group is written as $\mathbb{I}(\psi_i,\theta_i,\theta_{i-1})$, the invariance is equal with Eq.(17).

⑷Cycle-symmetry: The symmetry transformation of mechanisms is replacing a parameter p □by $p+kT$, whose transformation group is integer additive group, written as $\mathbb{Z}(p)$. There is symmetry with 2π circle for motion parameters of planar mechanisms. And the dual transformation of all parameters of spherical mechanisms is identical transformation.

⑸ Permutation symmetry: If permutating certain mechanism parameters is the symmetry transformation of mechanisms, these parameters have permutation symmetry, whose transformation group is a permutation group. The constraint equations of planar single-loop mechanisms are vector summation, which have nothing to do with the order of vectors. If the same permutation is executed at the same time for structural parameters ($l_1,\ldots,l_N$) and motion parameters ($\theta_1,\ldots,\theta_N$), the constraint equations maintain invariant, and the transformation group is N-element symmetry group, whose dual transformation is identical transformation. The dual transformation of division conditions of structural space and solution space is always certain permutation.

For spherical mechanisms, it should be verified that the symmetry transformation of exchanging adjacent links *i* and *i*+1 is $\psi_i \leftrightarrow \psi_{i+1}$, ϕ_{i-1} and ϕ_{i+1} are replaced by $\phi_{i-1}+\Delta\phi$ and $\phi_{i+1}-\Delta\phi$, and $\Delta\phi$ can be solved, whose transformation process is shown in Fig.3. The steps of transformation are as follows,

$$\begin{aligned} &\boldsymbol{\Psi}_{i+1}\boldsymbol{\Phi}_i\boldsymbol{\Psi}_i = \boldsymbol{\Phi}\boldsymbol{\Psi}\boldsymbol{\Phi}' \quad \Leftrightarrow \quad \boldsymbol{\Psi}_{i+1}^{-1}\boldsymbol{\Phi}_i^{-1}\boldsymbol{\Psi}_i^{-1} = \boldsymbol{\Phi}^{-1}\boldsymbol{\Psi}^{-1}\boldsymbol{\Phi}'^{-1} \\ &\boldsymbol{\Psi}_i\boldsymbol{\Phi}_i\boldsymbol{\Psi}_{i+1} = \boldsymbol{\Phi}'\boldsymbol{\Psi}\boldsymbol{\Phi} \quad \Leftrightarrow \quad \boldsymbol{\Phi}\boldsymbol{\Phi}'^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Phi}_i\boldsymbol{\Psi}_{i+1}\boldsymbol{\Phi}_1^{-1}\boldsymbol{\Phi}' = \boldsymbol{\Phi}\boldsymbol{\Psi}\boldsymbol{\Phi}' \\ &\boldsymbol{\Phi}(\phi-\phi')\boldsymbol{\Psi}_i\boldsymbol{\Phi}_i\boldsymbol{\Psi}_{i+1}\boldsymbol{\Phi}(\phi'-\phi) = \boldsymbol{\Phi}\boldsymbol{\Psi}\boldsymbol{\Phi}' \end{aligned} \tag{17}$$

By composite transformation for the exchange of any adjacent links, the conclusion can be obtained that all permutations are the symmetry transformation of spherical mechanisms, whose transformation group is *N*-element symmetry group.

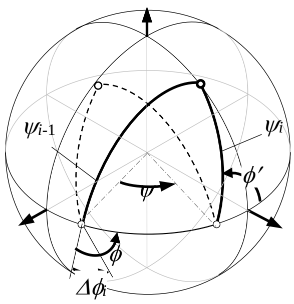


Fig.3 The exchange transformation of adjacent links

All above transformation groups are the subgroup of mechanism transformation groups, the product of all transformation groups forms mechanism transformation group. In the global analysis of mechanism kinematic characteristics, the structural space is divided to realize dimension classification firstly. Then the solution space is divided to obtain the global distribution of mechanism kinematic characteristics. So the symmetry of structural space should contain both the symmetry of structural parameters itself and the symmetry of structural parameters associated with motion parameters. The symmetry of solution space only contains the symmetry of motion parameters themselves.

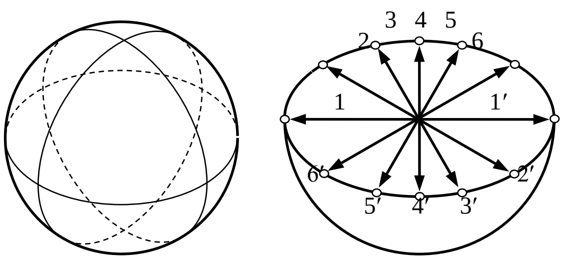
In planar single-loop mechanisms, the transformation groups of structural space are the direct product $\mathbb{R}^{+}_{M}(\boldsymbol{v})\otimes\mathbb{I}(\boldsymbol{v})\otimes\mathbb{S}(\boldsymbol{v})$ of the transformation groups of projective, inversion and permutation. The transformation groups of solution space are the direct product $\mathbb{R}_{A}(\boldsymbol{u})\otimes\mathbb{Z}(p)\otimes\mathbb{I}(\boldsymbol{u})$ of the transformation groups of rotation, circle and mirror. In spherical single-loop mechanisms, the transformation groups of structural space are the direct product $\mathbb{I}(\boldsymbol{v})\otimes\mathbb{Z}(p)\otimes\mathbb{S}(\boldsymbol{v})$ of the transformation groups of projective, circle and permutation of structural space, and the transformation groups of solution space are the direct product $\mathbb{I}(\boldsymbol{u})$ of the transformation groups of mirror and circle.

4. Applications of mechanism transformation group

Applications of the mechanism transformation group are very wide in global analysis of mechanism performance. Here only applications in three aspects for basic areas of parameter space, classification of division geometries and the generation of division conditions are introduced.

⑴ The basic area of parameter space: The basic area is a continuous set in parameter space, which can cover the whole parameter space under the action of mechanism transformation group, and the measure of overlapping areas is 0. Only dividing basic areas can realize global analysis of mechanism performance. The transformation group of mechanisms is decomposed into some direct products of subgroups, by using different subgroups to reduce parameter space, the basic area of mechanisms is finally obtained.

If the transformation subgroup of mechanisms is a D-dimensional continuous group, D-dimensions can be reduced for the dimension of parameter space. The whole motion transformation group is a 6-dimensional continuous group, which can make the dimension of motion space reducing 6-dimensions at most. The whole motion transformation group $\mathbb{R}_A(\boldsymbol{u})$ of planar single-loop mechanisms is a 1-dimensional rotation group, so the dimension of motion space can reduce 1. The projective transformation group of mechanisms is 1-dimension, so the dimension of parameter space can also reduce 1. The projective transformation group of planar single-loop mechanisms can reduce structural space to the unit sphere $l^2_1+\ldots+l^2_N=1$. If a transformation group is $\mathbb{R}^+_M(\boldsymbol{v})$, the reduced area is the whole sphere. While a transformation group is $\mathbb{R}_M(\boldsymbol{v})=\{1,-1\}\otimes\mathbb{R}^+_M(\boldsymbol{v})$, the reduced area is any half sphere, and points of symmetry transformation on the boundary should be bonded to a closed Mobius strip. The status of 3-dimensional structural space is shown in Fig.4. The unit sphere can also be replaced by other convex surfaces, such as a regular polyhedron with 2^N, $|l_1|+\ldots+|l_N|=1$.



(a) $\mathbb{R}^+_M(\boldsymbol{v})$ (b) $\mathbb{R}_M(\boldsymbol{v})$

Fig.4 The projective symmetry for reducing a three-dimensional space

If the transformation subgroup $\mathbb{H}$ of mechanisms is an infinite discrete group, after reducing, the dimension does not change, and analysis area reduces to a closed finite area from an infinite open area. For example, parameter set (p_1,…, p_N) with cycle symmetry makes the analysis area reducing to a N-dimensional torus from infinite area. Cases of 1-dimension and 2-dimension are shown in Fig.5. The motion space of planar single-loop mechanisms, and the motion space and structural space of spherical single-loop mechanisms belong to this case.

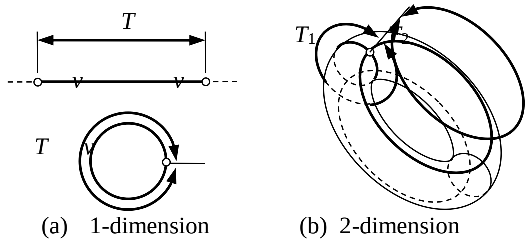


Fig.5 The analysis area after reducing the cycle symmetry of 1and 2-dimension space

If the transformation subgroup $\mathbb{H}$ of mechanisms is a finite discrete group, after reduction, the dimension keeps the same, but the measure of analysis area reduces to 1/|H|. The simplest transformation group, namely the inversion transformation group of single parameter $\mathbb{H}(x_1)$={1,-1}x_1, reduces the analysis area to a half, generally taking the range of x_1>0. The inversion transformation group of N parameter is $\mathbb{H}(x_1)\otimes\ldots\otimes\mathbb{H}(x_N)$, whose analysis basic area reduces to $1/2^N$, generally taking the first quadrant, x_i>0. The joint inversion transformation group of N parameter is $\mathbb{H}(\mathbf{x})$={1,-1}(x_1,…, x_N), whose analysis basic area is reduced to be a half, taking any half is feasible. But points of symmetry transformation on the boundary should be bonded. The case of 2 parameters is shown in Fig.Fig. 1(a).

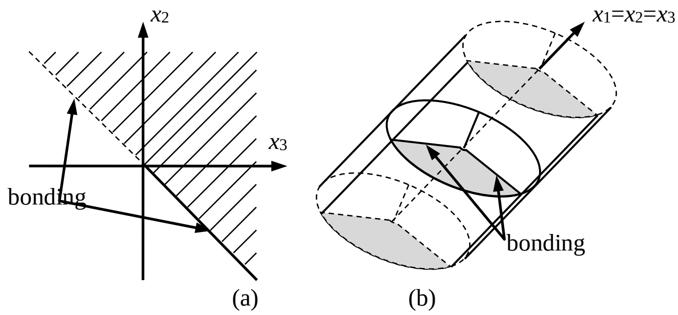


Fig. 1 (a) The inversion symmetry of 2 parameters and (b) the cycle symmetry of 3 parameters

The simplest permutation group is the symmetry group $\mathbb{S}(x_1,x_2)$of 2 parameters, whose analysis area is reduced to be a half, generally taking the range of x_1>x_2. The transformation group is exchanging two parameter sets (x_1,…,x_N) and (y_1,…,y_N), whose analysis area is reduced to be a half. With constructing any function satisfied $f(x_1,\dots,x_N;y_1,\dots,y_N)= -f(x_1,\dots,x_N;y_1,\dots,y_N;x_1,\dots,x_N)$, the range of $f>0$ is taken, and symmetry transformation exchanging points on the boundary are bonded. When the transformation group is the symmetry group of N-parameters, whose analysis area is reduced to be $1/N!$, generally taking the range of $x_i>x_{i+1}$. While if the transformation group is the cyclic group of N-parameters, whose analysis area is reduced to be $1/N$. With constructing any function $f(x_1,\dots,x_N)$ satisfied $f(x,\dots,x)=0$, the range of $f(x_1,\dots,x_N)>0$ and $f(x_2,\dots,x_N,x_1)<0$ is taken as the analysis area, and symmetry transformation points on the boundary are bonded. The case of cyclic transformation group of 3 parameters is shown in Fig.Fig. 1(b).

According to above analysis, the basic analysis area of structural space and solution space of planar single-loop mechanisms can be determined,

$$\begin{array}{lll} l_i \ge 0 & & \\ l_i - l_{i+1} \ge 0 & 0 \le \theta_1 < \pi & \\ l_1 + \cdots + l_N = 1 & 0 \le \theta_i < 2\pi & i \ne 1 \end{array} \quad (18)$$

The basic area of structural space and solution space of spherical single-loop mechanisms is,

$$\begin{array}{lll} 0 \le \psi_i < \pi/2 & 0 \le \phi_1 < \pi & \\ \psi_i - \psi_{i+1} \ge 0 & 0 \le \phi_i < 2\pi & i \ne 1 \end{array} \quad (19)$$

(2) The classification of division geometries

The global analysis of mechanism performance can also be carried out through following steps. Firstly the whole parameter space is divided. Then the symmetry of division results is analyzed. Finally correct analysis results can be obtained. The division results have two equivalent forms of expressions, with one being sign sequence, and the other being complex expression. The sign sequence of structural space is three value sequence $\boldsymbol{S}^v=(s^v{}_1,\dots,s^v{}_{Nv})$, whose length is equal with the number N_v of division conditions, and their elements are

$$s_i^v = \begin{cases} 1 & f_i^v > 0 \\ 0 & f_i^v = 0 \\ -1 & f_i^v < 0 \end{cases} \quad (21)$$

Any geometry is always corresponding to only sign sequences $\boldsymbol{S}^v$, where there may exist 3^{Nv} sign sequences, but some sign sequences have no corresponding geometries. The definition of sign sequences $\boldsymbol{S}^u=(s^u{}_1,\dots, s^u{}_{Nu})$ of solution space is the same with this.

After solution division, all coordinates of 0-dimensional simplex (namely vertex) is sorted into a $N_0\times N_v$ coordinate matrix A, where N_0, is the number of vertexes. Then any geometry $\boldsymbol{P}^v$ is the complex of structural space, which is expressed by a vertex sequence,

$$\boldsymbol{P}^v = < p_1^v,\cdots,p_{N_P^v}^v > \qquad p_i^v \in \{1,2,\cdots,N_0\} \quad (22)$$

The coordinate matrix of geometry $\boldsymbol{P}^v$ is a $N_p \times N_v$ matrix. There is large number of vertex combinations, but most of them is not real geometries. In the solution space, complex $\boldsymbol{P}^u$ and its coordinate matrix $\boldsymbol{A}^u{}_P$ are same with this.

If $\mathbb{G}$ is the transformation group of mechanisms, and $\mathbb{H}$ is the coordinate matrix $\boldsymbol{A}_P$ of geometry $\boldsymbol{P}$, satisfied $\mathbb{H}(\boldsymbol{A}_P)=\boldsymbol{A}_P$, the congeneric homogeneous geometry of $\boldsymbol{P}$ can be obtained by the left coset of group $\mathbb{H}$ in group $\mathbb{G}$. The transformation group of congeneric geometry $g(\boldsymbol{A}_P)$ is $g\mathbb{H}g^{-1}$. The number of congeneric homogeneous geometries is equal to the index of group $\mathbb{H}$ in group $\mathbb{G}$, namely $|G|/|H|$. Homogeneous geometries have same properties. The classification of geometries can also be equivalently carried out through the dual transformation group.

Generally, the symmetry analysis of division results of high-dimensional parameter space is complex and counterintuitive, which is suitable for computer processing with a fixed program. Thus, the division of structural space of planar mechanisms with 3-parameters is taken as an example to illustrate here, although the corresponding triangle does not form mechanisms, which can make the process of division and symmetry analysis clearly.

As shown in Fig.7, projective symmetry has reduced the division of the unit sphere. Four independent division conditions are $\{f_1{}^v=l_1+l_2+l_3=0,\ f_2{}^v=-l_1+l_2+l_3=0,\ f_3{}^v=l_1-l_2+l_3=0,\ f_4{}^v=l_1+l_2-l_3=0\}$, whose transformation group can be decomposed into direct product of permutation symmetry group $\mathbb{S}_3(\boldsymbol{l})$ and inversion transformation group with 3-parameters, namely $\mathbb{G}=\mathbb{S}_3(\boldsymbol{l})\otimes\mathbb{I}(l_1)\otimes\mathbb{I}(l_2)\otimes\mathbb{I}(l_3)$, with the order of $|G|=6\times2\times2\times2=48$. Although there are 14 areas, 24 lines and 12 vertexes in the division results, symmetry analysis indicates that there are only two kinds of areas with different characters, one kind of boundary and one kind of vertex. The information of complex expression, sign sequence and transformation group for these typical representative types of geometries is listed in Table.1.

Tab.1 The information for the typical representative types of geometries

	Complex expression	Coordinate matrix	sign sequence	transformation group		number of geometry	Characters of mechanisms
				Group $\mathbb{G}$	$\|\mathbb{G}\|$		
area Ⅰ	<1,2,3>	$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\frac{1}{\sqrt{3}}$	(1,1,1,1)	$\mathbb{S}_3(l_1,l_2,l_3)$	6	8	Normal triangles
area Ⅱ	<2,3,4,5>	$\begin{bmatrix}1&0&1\\0&1&1\\-1&0&1\\0&-1&1\end{bmatrix}\frac{1}{\sqrt{3}}$	(1,1,1,-1)	$\mathbb{S}_2(l_1,l_2)\otimes\mathbb{I}(l_1)\otimes\mathbb{I}(l_2)$	8	6	don't closed
boundary	<2,3>	$\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\frac{1}{\sqrt{3}}$	(1,1,1,0)	$\mathbb{S}_2(l_1,l_2)$	2	24	boundary between Area Ⅰ and Ⅱ
vertex	<1>	$\begin{bmatrix}1&1&0\end{bmatrix}\frac{1}{\sqrt{3}}$	(1,0,0,-1)	$\mathbb{S}_2(l_1,l_2)\otimes\mathbb{I}(l_1)$	4	12	

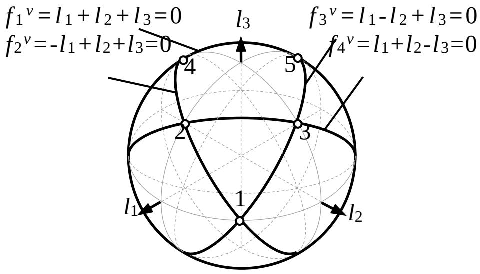


Fig. 2 The division of structural space with 3-parameters

(3)The classification of division conditions

If G is the transformation group of mechanism constraint equations , the transformation group of universal set ***T*** of division conditions of mechanism parameter space s is also $\mathbb{G}$. The transformation group for one division condition t is the subgroup $\mathbb{H}(t)$ of $\mathbb{G}$. The cognate division condition of t is given by the coset of $\mathbb{H}(t)$ in $\mathbb{G}$, whose number is $k = |G|/|H|$.

All division conditions are divided into some equivalence classes. Let t be a division condition, the equivalence classes of t are generated by the coset of transformation group of t in $\mathbb{G}$. On the other hand, by using the division condition in identical equivalence classes to divide parameter space, the division results are equivalent. If known the division results of parameter space divided by t, the division results of division conditions in the equivalence classes of t can be obtained by the coset transformation.

The division condition of structural space $l_1+\ldots+l_N=0$ of planar single-loop mechanisms is taken as an example to be analyzed, whose transformation group is the direct product of symmetry group and the protective transformation group of structural parameters, namely $\mathbb{S}_N(l_1,\ldots,l_N)\otimes \mathbb{R}^+_M(l_1,\ldots,l_N)$.

The direct product of symmetry group and protective transformation group with the N-elements structural parameter of transformation group is $\mathbb{S}_N(l_1,\ldots,l_N)\otimes\mathbb{R}^+_M(l_1,\ldots,l_N)$. Compared with the transformation group of universal set of division conditions, the number of its coset is 2^N. All division conditions can be generated by the inversion transformation group $\mathbb{I}(l_1)\otimes\ldots\otimes\mathbb{I}(l_1)$ of structural parameters, and only belong to one type. The case of spherical single-loop mechanisms is similar with this.

5. Conclusions

In this paper, the group theory is introduced to global characteristics analysis of mechanisms. The basic contents include three main parts. The first is transformation group and its basic properties. The second is the definition of transformation group of mechanisms. The third is the application of mechanism transformation group in global characteristics analysis of mechanisms. By using group theory as a mathematical tool for symmetry properties of mechanism parameters, the analytical range of parameters space is obviously narrowed, so calculation workload can be greatly reduced, at the same time the incorrect classification results can be avoided. The group theory is also a necessary mathematical tool for the dimension classification and characteristics analysis of mechanisms.

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