THE NORMS AND THE LOWER BOUNDS FOR MATRIX OPERATORS ON WEIGHTED DIFFERENCE SEQUENCE SPACES

Davoud FOROUTANNIA[1], Hadi ROOPAEI[2]

This paper is concerned with the problem of finding upper bounds and lower bounds of matrix operators from $l_p(v)$ into $l_p(w,\Delta)$, where (v_n) and (w_n) are two non-negative sequences. Moreover, the norms and lower bounds of matrix operators such as quasi-summability matrices and Hilbert operator are computed.

Keywords: Matrix operator, Norm, Lower bound, Quasi-summability matrix, Hilbert matrix, Weighted sequence space.

MSC2010: 26D15, 40C05, 40G05, 47B37.

1. Introduction

Let $p \geq 1$ and ω denote the set of all real-valued sequences. The space l_p is the set of all real sequences $x = (x_n) \in \omega$ such that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

If $w = (w_n) \in \omega$ is a non-negative sequence, we define the weighted sequence space $l_p(w)$ as follows:

$$l_p(w) := \left\{ x = (x_n) \in \omega \ : \ \sum_{n=1}^{\infty} w_n |x_n|^p < \infty \right\},$$

with norm, $\|.\|_{p,w}$, which is defined in the following way:

$$\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p\right)^{1/p}.$$

The idea of difference sequence spaces was introduced by Kizmaz [9]. Similarly, we define the sequence space $l_p(w,\Delta)$ as below:

$$l_p(w,\Delta) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} w_n |x_n - x_{n+1}|^p < \infty \right\},$$

with semi-norm, $\|.\|_{p,w,\Delta}$, which is defined by

$$\|x\|_{p,w,\Delta} = \left(\sum_{n=1}^{\infty} w_n |x_n - x_{n+1}|^p\right)^{\frac{1}{p}}.$$

[1]Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran, E-mail: foroutan@vru.ac.ir

[2]Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran, E-mail: h.roopaei@gmail.com

Note that this function will be not a norm, since if $x = (1, 1, 1, ...)$ then $\|x\|_{p,w,\Delta} = 0$ while $x \neq 0$. It is significant that in the special case $w_n = 1$ for all n, we have $l_p(w) = l_p$ and $l_p(w, \Delta) = l_p(\Delta)$.

Let (v_n) and (w_n) be two non-negative sequences. In this paper, we shall consider the inequality of the form

$$\|Ax\|_{p,w,\Delta} \leq U\|x\|_{p,v},$$

for all sequence $x \in l_p(v)$. The constant U is not depending on x, and we seek the smallest possible value of U. We write $\|A\|_{p,v,w,\Delta}$ for the norm of A as an operator from $l_p(v)$ into $l_p(w, \Delta)$, and $\|A\|_{p,w,\Delta}$ for the norm of A as an operator from $l_p(w)$ into $l_p(w, \Delta)$, and $\|A\|_{p,\Delta}$ for the norm of A as an operator from l_p into $l_p(\Delta)$, and $\|A\|_p$ for the norm of A as an operator from l_p into itself.

The problem of finding the upper bounds of certain matrix operators on the sequence spaces $l_p(w)$, $d(w, p)$ and bv_p are studied before in [5], [6], [8] and [10]. In the study, we examine this problem for matrix operators from $l_p(v)$ into $l_p(w, \Delta)$ and we consider certain matrix operators such as quasi-summability matrices and Hilbert operator.

Let A be a matrix operator with non-negative entries from $l_p(v)$ into $l_p(w, \Delta)$. The other purpose of this study is to consider the inequality of the form

$$\|Ax\|_{p,w,\Delta} \geq L\|x\|_{p,v},$$

for all non-negative decreasing sequence $x \in l_p(v)$, where L is a constant not depending on x. Also we seek the largest possible value of L.

The problem of finding the lower bounds of matrix operators was introduced by Lyons [11], and has been intensively studied on l_p by Bennett [1,2,3]. Jameson [6] was computed the lower bounds of operators on Lorentz sequence space $d(w, 1)$. Then Jameson and Lashkaripour [7] were examined lower bounds of certain matrix operators on $l_p(w)$ and $d(w, p)$. More recently, this problem has been developed in the block sequence space [5]. In this paper, we study the problem of finding the lower bound for matrix operators from $l_p(v)$ into $l_p(w, \Delta)$ and investigate certain matrix operators such as quasi-summability matrices and Hilbert operator.

2. The norm of matrix operators from $l_1(v)$ into $l_1(w, \Delta)$

In this section, we tend to compute the norm of operators from $l_1(v)$ into $l_1(w, \Delta)$. We may begin with the following theorem which is essential in the study.

Theorem 2.1. *Let $A = (a_{n,k})$ be a matrix operator and (v_n), (w_n) be two non-negative sequences. If $\sup_k \frac{u_k}{v_k} < \infty$ where $u_k = \sum_{n=1}^{\infty} w_n |a_{n,k} - a_{n+1,k}|$ for all k, then A is a bounded operator from $l_1(v)$ into $l_1(w, \Delta)$ and*

$$\|A\|_{1,v,w,\Delta} = \sup_n \frac{u_n}{v_n}.$$

In particular if $v_n = w_n = 1$ for all n, then A is a bounded operator from l_1 into $l_1(\Delta)$ and $\|A\|_{1,\Delta} = \sup_n u_n$.

Proof. Let $M = \sup_n \frac{u_n}{v_n}$ and (x_n) be in $l_1(v)$. We have

$$\begin{aligned}\|Ax\|_{1,w,\Delta} &= \sum_{n=1}^{\infty} w_n \left|\sum_{k=1}^{\infty}(a_{n,k}-a_{n+1,k})x_k\right| \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} w_n|a_{n,k}-a_{n+1,k}||x_k| \\ &= \sum_{k=1}^{\infty} u_k|x_k| \leq M\sum_{k=1}^{\infty} v_k|x_k| = M\|x\|_{1,v},\end{aligned}$$

which says that $\|A\|_{1,v,w,\Delta} \leqslant M$. Conversely, we take $x = e_n$ which e_n denotes the sequence having 1 in place n and 0 elsewhere, then $\|x\|_{1,v} = v_n$ and $\|Ax\|_{1,w,\Delta} = u_n$ which proves that $\|A\|_{1,v,w,\Delta} = M$. □

We say that $A = (a_{n,k})$ is a quasi-summability matrix if it is an upper-triangular matrix, i.e. $a_{n,k} = 0$ for $n > k$, and $\sum_{n=1}^{k} a_{n,k} = 1$ for all k. In the following, we consider the norm problem for quasi-summability matrix operators.

Theorem 2.2. *Let $A = (a_{n,k})$ be an upper-triangular matrix with non-negative entries and (w_n) be an increasing sequence. If the columns of A are decreasing, i.e.*

$$a_{n,k} \geq a_{n+1,k}, \qquad (n,k = 1,2,\cdots)$$

and $M = \sup_k a_{1,k} < \infty$, then A is a bounded operator from $l_1(w)$ into $l_1(w,\Delta)$ and

$$\|A\|_{1,w,\Delta} \leq M.$$

In particular if A is a quasi-summability matrix, then $\|A\|_{1,w,\Delta} = 1$.

Proof. According to above notation

$$u_k = \sum_{n=1}^{k-1} w_n(a_{n,k} - a_{n+1,k}) + w_k a_{k,k}.$$

Since the sequences (w_n) is increasing

$$\frac{u_k}{w_k} \leq \sum_{n=1}^{k-1}(a_{n,k} - a_{n+1,k}) + a_{k,k} = a_{1,k},$$

so by applying Theorem 2.1, we have $\|A\|_{1,w,\Delta} \leq M$. In particular if A is a quasi-summability matrix, we deduce that $\|A\|_{1,w,\Delta} \leq 1$. Using the fact that $\|Ae_1\|_{1,w,\Delta} = \|e_1\|_{1,w} = w_1$ finishes the proof. □

Next, we identify a class of quasi-summability matrices for which the norm problem is very easy. If (a_n) is a non-negative sequence with $a_1 > 0$ and $A_n = a_1 + \cdots + a_n$, the matrix $M_a = (a_{n,k})$ is defined as follows:

$$a_{n,k} = \begin{cases} \frac{a_n}{A_k} & n \leq k \\ 0 & n > k. \end{cases} \tag{1}$$

M_a is the transpose of the weighted mean matrix.

Corollary 2.1. *If (a_n) is decreasing and (w_n) is increasing, then M_a is a bounded operator from $l_1(w)$ into $l_1(w,\Delta)$ and*

$$\|M_a\|_{1,w,\Delta} = 1.$$

Note that M_a is called the Copson matrix when $a_n = 1$ for all n, hence the Copson Matrix $C = (c_{n,k})$ defined by

$$c_{n,k} = \begin{cases} \frac{1}{k} & for\ n \leq k \\ 0 & for\ n > k. \end{cases}$$

Corollary 2.2. *Let C be the Copson operator and (v_n) and (w_n) be two non-negative sequences. If $\sup_k \frac{w_k}{kv_k} < \infty$, then C is a bounded operator from $l_1(v)$ into $l_1(w, \Delta)$ and*

$$\|C\|_{1,v,w,\Delta} = \sup_k \frac{w_k}{kv_k}.$$

Proof. Since

$$u_k = \sum_{n=1}^{\infty} w_n(c_{n,k} - c_{n+1,k}) = \frac{w_k}{k},$$

by applying Theorem 2.1, we obtain the desired result. □

In the next statement, we try to compute the norm of the certain matrix operators from l_1 into $l_1(\Delta)$.

Theorem 2.3. *Suppose that $A = (a_{n,k})$ is a matrix with non-negative entries and $M = \sup_k a_{1,k} < \infty$. If the columns of A are decresing i.e.*

$$a_{n,k} \geq a_{n+1,k}, \qquad (n, k = 1, 2, \cdots)$$

and $\lim_{n\to\infty} a_{n,k} = 0$, for all k. Then A is a bounded operator from l_1 into $l_1(\Delta)$ and

$$\|A\|_{1,\Delta} = M.$$

In particular if A is a quasi-summability matrix, then $\|A\|_{1,\Delta} = 1$.

Proof. Since

$$u_k = \sum_{n=1}^{\infty} (a_{n,k} - a_{n+1,k}) = a_{1,k},$$

by using Theorem 2.1, we obtain the desired result. □

We recall the Hilbert operator H which is defined by the matrix:

$$h_{n,k} = \frac{1}{n+k}, \qquad (n, k = 1, 2, \cdots).$$

Corollary 2.3. *If H is the Hilbert operator, then H is a bounded operator from l_1 into $l_1(\Delta)$ and $\|H\|_{1,\Delta} = \frac{1}{2}$.*

In the following, we try to solve the problem of finding the norm of the Hilbert matrix operator from $l_1(v)$ into $l_1(w, \Delta)$. For this purpose, the same as the most studies of the Hilbert operator, it uses the well-known integral

$$\int_0^{\infty} \frac{1}{t^{\alpha}(t+c)} dt = \frac{\pi}{c^{\alpha} \sin \alpha\pi},$$

where $0 < \alpha < 1$, (see [4], page 285).

Theorem 2.4. *Let H be the Hilbert operator. If $w_n = \frac{1}{n^{\alpha}}$ for all n, where $0 < \alpha < 1$, then H is a bounded operator from $l_1(w)$ into $l_1(w, \Delta)$ and*

$$\|H\|_{1,w,\Delta} \leq \frac{\pi}{\sin \alpha\pi} \left(1 - \frac{1}{2^{\alpha}}\right).$$

Proof. According to above notation

$$\begin{aligned} u_n &= \sum_{i=1}^{\infty} \frac{1}{i^\alpha}\left(\frac{1}{i+n} - \frac{1}{i+n+1}\right) \leq \int_0^\infty \frac{1}{t^\alpha}\left(\frac{1}{t+n} - \frac{1}{t+n+1}\right) dt \\ &= \frac{\pi}{\sin \alpha\pi}\left(\frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha}\right), \end{aligned}$$

so

$$n^\alpha u_n \leq \frac{\pi}{\sin \alpha\pi}\left(1 - \left(\frac{n}{n+1}\right)^\alpha\right) \leq \frac{\pi}{\sin \alpha\pi}\left(1 - \frac{1}{2^\alpha}\right),$$

hence $\|H\|_{1,w,\Delta} \leq \frac{\pi}{\sin \alpha\pi}\left(1 - \frac{1}{2^\alpha}\right)$. □

3. Upper bounds of matrix operators from $l_p(v)$ into $l_p(w, \Delta)$

In this section the problem of finding the norm of certain matrix operators such as the transpose of the weighted mean, Copson and Hilbert from $l_p(v)$ into $l_p(w, \Delta)$ are considered. We first give the Schur's Theorem and a lemma which are essential in the study.

Lemma 3.1 ([8], Lemma 2.2)**.** *Let $p > 1$ and $B = (b_{n,k})$ be a matrix operator with $b_{n,k} \geq 0$ for $n, k = 1, 2, \cdots$. Suppose that (s_n) and (t_k) are two sequences of strictly positive numbers such that for some C, R*

$$s_n^{1/p} \sum_{k=1}^{\infty} b_{n,k} t_k^{-1/p} \leq R \quad (for\ n \geq 1), \qquad t_k^{(p-1)/p} \sum_{n=1}^{\infty} b_{n,k} s_n^{(1-p)/p} \leq C \quad (for\ k \geq 1).$$

Then $\|B\|_p \leq R^{(p-1)/p} C^{1/p}$.

Lemma 3.2. *Let $p \geq 1$ and (v_n), (w_n) be two non-negative sequences. If $A = (a_{n,k})$ and $B = (b_{n,k})$ are two matrix operators such that $b_{n,k} = \left(\frac{w_n}{v_k}\right)^{1/p} (a_{n,k} - a_{n+1,k})$, then*

$$\|A\|_{p,v,w,\Delta} = \|B\|_p.$$

Hence, if B is a bounded operator on l_p, then A will be a bounded operator from $l_p(v)$ into $l_p(w, \Delta)$.

Proof. For every $x \in l_p(v)$, we define $y = (y_k)$ by $y_k = v_k^{1/p} x_k$. It is obvious that $\|x\|_{p,v} = \|y\|_p$, and

$$\begin{aligned} \|A\|_{p,\Delta,w}^p &= \sup_{x \in l_p(v)} \frac{\|Ax\|_{p,\Delta,w}^p}{\|x\|_{p,v}^p} = \sup_{x \in l_p(v)} \frac{\sum_{n=1}^\infty w_n \left|\sum_{k=1}^\infty (a_{n,k} - a_{n+1,k}) x_k\right|^p}{\sum_{k=1}^\infty v_k |x_k|^p} \\ &= \sup_{y \in l_p} \frac{\sum_{n=1}^\infty \left|\sum_{k=1}^\infty \frac{w_n^{1/p}(a_{n,k} - a_{n+1,k})}{v_k^{1/p}} y_k\right|^p}{\sum\limits_{k=1}^\infty |y_k|^p} = \sup_{y \in l_p} \frac{\|By\|_p^p}{\|y\|_p^p} = \|B\|_p^p. \end{aligned}$$

□

In the following, we investigate the norm of the transpose of the weighted mean matrix.

Theorem 3.1. *Let $p > 1$ and (a_n) be a decreasing sequence such that $a_1 = a_2 = 1$ and $\lim_{n\to\infty} A_n = \infty$, where $A_n = \sum_{i=1}^n a_i$. If M_a is defined as in (1) and $v_n = \left(\frac{A_{n-1}}{a_n}\right)^p$ for all n, with $A_0 = 1$, then M_a is a bounded operator from $l_p(v)$ into $l_p(\Delta)$ and*

$$\|M_a\|_{p,v,1,\Delta} = 1.$$

Proof. By applying Lemma 3.2 we have $\|M_a\|_{p,v,1,\Delta} = \|B\|_p$, where

$$b_{n,k} = \begin{cases} \frac{a_k(a_n - a_{n+1})}{A_{k-1}A_k} & for\ n < k \\ \frac{a_k^2}{A_{k-1}A_k} & for\ n = k \\ 0 & for\ n > k. \end{cases}$$

In Lemma 3.1, take $s_n = t_n = 1$ and let C, R be defined as before. Since

$$\begin{aligned} \sum_{k=1}^{\infty} b_{n,k} &= \frac{a_n^2}{A_{n-1}A_n} + \sum_{k=n+1}^{\infty} \frac{a_k(a_n - a_{n+1})}{A_{k-1}A_k} \\ &= \frac{a_n^2}{A_{n-1}A_n} + (a_n - a_{n+1}) \sum_{k=n+1}^{\infty} \left(\frac{1}{A_{k-1}} - \frac{1}{A_k}\right) = \frac{a_n^2}{A_{n-1}A_n} + \frac{a_n - a_{n+1}}{A_n}, \end{aligned}$$

we have $\sum_{k=1}^{\infty} b_{1,k} = 1$ and

$$\sum_{k=1}^{\infty} b_{n,k} = \frac{a_n}{A_{n-1}} - \frac{a_{n+1}}{A_n},$$

for $n > 1$, so $R \leq 1$. Also, since

$$\sum_{n=1}^{\infty} b_{n,k} = \sum_{n=1}^{k-1} \frac{a_k(a_n - a_{n+1})}{A_{k-1}A_k} + \frac{a_k^2}{A_{k-1}A_k} = \frac{a_1 a_k}{A_{k-1}A_k} \leq 1,$$

we deduce that $C \leq 1$, so $\|M_a\|_{p,v,1,\Delta} \leq 1$. Now let $x = (1, 0, 0, \cdots)$, we have $\|x\|_{p,v} = 1$ and $\|M_a x\|_{p,\Delta} = 1$. So $\|M_a\|_{p,v,1,\Delta} \geq 1$, this completes the proof of the theorem. □

Now we are ready to compute the norm of the Copson matrix operator.

Theorem 3.2. *Suppose that $p > 1$ and (v_n), (w_n) are two non-negative sequences. If C is the Copson matrix operator and*

$$M = \sup_n \frac{1}{n} \left(\frac{w_n}{v_n}\right)^{1/p} < \infty,$$

then $\|C\|_{p,v,w,\Delta} = M$. In particular if $v_n = w_n$ for all n, we have $\|C\|_{p,w,\Delta} = 1$.

Proof. By applying Lemma 3.2 we have $\|C\|_{p,v,w,\Delta} = \|B\|_p$, where

$$b_{n,k} = \begin{cases} \frac{1}{n}\left(\frac{w_n}{v_n}\right)^{1/p} & for\ n = k \\ 0 & otherwise. \end{cases}$$

In Lemma 3.1, take $s_n = t_n = 1$ and let C, R be defined as before. Since the matrix B is diagonal, we deduce that $R \leq M$ and $C \leq M$, so $\|C\|_{p,v,w,\Delta} \leq M$. Now let $x = e_n$, we have $\|x\|_{p,v} = v_n^{1/p}$ and $\|Cx\|_{p,w,\Delta} = w_n^{1/p}/n$. So $\|C\|_{p,v,w,\Delta} \geq M$, which concludes the proof. □

Finally, we consider the norm of the Hilbert matrix operator.

Theorem 3.3. *Let H be the Hilbert operator and $p > 1$. If $w_n = \frac{1}{n^\alpha}$ for all n, where $1 - p < \alpha < 1$, then H is a bounded operator from $l_p(w)$ into $l_p(w, \Delta)$ and*

$$\|H\|_{p,w,\Delta} \leq \frac{\pi}{\sin[(1-\alpha)\pi/p]} \left(1 - \frac{1}{2^{(1-\alpha)/p}}\right).$$

Proof. By applying Lemma 3.2 we have $\|H\|_{p,w,\Delta} = \|B\|_p$, where

$$b_{n,k} = \left(\frac{k}{n}\right)^{\alpha/p} \left(\frac{1}{n+k} - \frac{1}{n+k+1}\right).$$

In Lemma 3.1, take $s_n = t_n = n$, and let C, R be defined as before. Then

$$b_{n,k} s_n^{1/p} t_k^{-1/p} = \left(\frac{1}{n+k} - \frac{1}{n+k+1}\right) \left(\frac{n}{k}\right)^{(1-\alpha)/p}.$$

Write $M = \frac{\pi}{\sin[(1-\alpha)\pi/p]} \left(1 - \frac{1}{2^{(1-\alpha)/p}}\right)$, it follows that $R \leq M$ and similarly $C \leq M$. Hence $\|H\|_{p,w,\Delta} \leq M$, which proves the theorem. □

Theorem 3.4. *Let H be the Hilbert operator and $p > 1$. If $v_n = \frac{1}{n^{p+\alpha}}$ and $w_n = \frac{1}{n^\alpha}$ for all n, where $1/p - 1 < p + \alpha < 1/p$. Then H is a bounded operator from $l_p(v)$ into $l_p(w, \Delta)$ and*

$$\|H\|_{p,v,w,\Delta} \leq \frac{\pi}{\sin(\beta\pi)} \left(1 - \frac{1}{2^\beta}\right),$$

where $\beta = 1/p - p - \alpha$.

Proof. The proof is essentially same as that of Theorem 3.3 and so we omit the details. □

4. Lower bounds of matrix operators from $l_p(v)$ into $l_p(w, \Delta)$

In this part of the study, we consider the lower bound, L, of the form

$$\|Ax\|_{p,w,\Delta} \geq L\|x\|_{p,v},$$

for all non-negative decreasing sequence x. The constant L is not depending on x and we seek the largest possible value of L. We are looking for the problem of finding the lower bound of certain matrix operators from $l_p(v)$ into $l_p(w, \Delta)$.

We begin with a lemma, which is the key to prove the main theorem of this section.

Lemma 4.1 ([7], Lemma 2)**.** *Suppose $p \geq 1$ and sequences (a_i) and (x_i) are nonnegative, and that (x_i) is decreasing and tends to zero. If $A_n = \sum_{i=1}^n a_i$, $A_0 = 0$ and $B_n = \sum_{i=1}^n a_i x_i$, then*

(i) $B_n^p - B_{n-1}^p \geqslant (A_n^p - A_{n-1}^p) x_n^p$ for all n;

(ii) if $\sum_{i=1}^\infty a_i x_i$ is convergent, then

$$\left(\sum_{i=1}^\infty a_i x_i\right)^p \geqslant \sum_{n=1}^\infty A_n^p (x_n^p - x_{n+1}^p).$$

Theorem 4.1. *Let $p \geq 1$ and (v_n), (w_n) be non-negative sequences, and that $\sum_{n=1}^\infty v_n = \infty$. Also let $A = (a_{n,k})$ be a matrix operator from $l_p(v)$ into $l_p(w, \Delta)$ such that $a_{n,k} \geq a_{n+1,k}$ for all n, k. If $V_n = \sum_{k=1}^n v_k$ and $S_n = \sum_{i=1}^\infty w_i \left(\sum_{k=1}^n (a_{i,k} - a_{i+1,k})\right)^p$ then*

$$\|Ax\|_{p,w,\Delta} \geq L\|x\|_{p,v},$$

for all non-negative decreasing sequence $x \in l_p(v)$, where

$$L^p = \inf_n \frac{S_n}{V_n}.$$

This constant is the best possible.

Proof. Let x be in $l_p(v)$ such that $x_1 \geq x_2 \geq \cdots \geq 0$ and $\|x\|_{p,v} = 1$. The condition $\sum_{n=1}^{\infty} v_n = \infty$ implies that $\lim_{n\to\infty} x_n = 0$. Applying Lemma 4.1 and Able's identity, we have

$$\begin{aligned} \|Ax\|_{p,w,\Delta}^p &= \sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k}) x_k \right)^p \\ &\geq \sum_{n=1}^{\infty} w_n \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} (a_{n,k} - a_{n+1,k}) \right)^p (x_i^p - x_{i+1}^p) \\ &= \sum_{i=1}^{\infty} S_i (x_i^p - x_{i+1}^p) \geq L^p \sum_{i=1}^{\infty} V_i \left(x_i^p - x_{i+1}^p \right) = L^p \|x\|_{p,v}^p. \end{aligned}$$

So $\|Ax\|_{p,w,\Delta}^p \geq L^p \|x\|_{p,v}^p$. To show that the above constant is the best possible, we take $x_1 = x_2 = \cdots = x_n = 1$, and $x_k = 0$ for all $k \geq n+1$, then $\|x\|_{p,v}^p = V_n$ and $\|Ax\|_{p,w,\Delta}^p = S_n$, which finishes the proof of the theorem. □

The problem of finding lower bound for the Copson matrix operator for certain weights is solved in the following.

Theorem 4.2. *Let $p \geq 1$ and (v_n), (w_n) be non-negative sequences, and that $\sum_{n=1}^{\infty} v_n = \infty$. If C is the Copson operator, then*

$$\|Cx\|_{p,w,\Delta} \geq L \|x\|_{p,v},$$

for all non-negative decreasing sequence x, where

$$L^p = \inf_n \frac{w_1 + \frac{w_2}{2^p} + \cdots + \frac{w_n}{n^p}}{V_n}.$$

In particular
(i) if $V_n = n^{p+1}$ and $w_n = n^{2p}$ for all n, then $L^p = \frac{1}{p+1}$;
(ii) if $v_n = 1$ and $w_n = n^p$ for all n, then $L = 1$;
(iii) if $v_n = w_n$ for all n, then $L = 0$.

Proof. With the notation of Theorem 4.1, $S_n = \sum_{k=1}^{n} \frac{w_k}{k^p}$ which completes the proof of the first part. If $w_n = n^{2p}$ and $V_n = n^{p+1}$ then

$$L^p = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)^p + \left(\frac{2}{n}\right)^p + \cdots + \left(\frac{n}{n}\right)^p}{n} = \int_0^1 x^p dx = \frac{1}{p+1}.$$

The remaining of the proof is obvious. □

Write $t_n = \sum_{i=1}^{\infty} w_i (a_{i,n} - a_{i+1,n})^p$, since $s_n = S_n - S_{n-1}$ we have the following statement.

Proposition 4.1. *If v, w and A satisfy the conditions of Theorem 4.1, and*

$$a_{n,k} - a_{n+1,k} \geq a_{n,k+1} - a_{n+1,k+1},$$

for $n, k = 1, 2, \cdots$, then

$$L^p \geq \inf_n [n^p - (n-1)^p] \frac{t_n}{w_n}.$$

Proof. See Proposition 1 of [7]. □

Finally, we compute the lower bound of Hilbert operator from $l_p(w)$ into $l_p(\Delta, w)$.

Theorem 4.3. *Suppose that $w_n = \frac{1}{n^\alpha}$ and $v_n = \frac{1}{n^{p+\alpha}}$ for all n, where $0 \leq p + \alpha \leq 1$ and $p \geq 1$. If H is the Hilbert operator, then*

$$\|Hx\|_{p,w,\Delta} \geq L\|x\|_{p,v},$$

for all non-negative decreasing sequence x, where

$$L^p = \sum_{i=1}^{\infty} \frac{1}{i^\alpha (i+1)^p (i+2)^p}.$$

Proof. We have

$$\begin{aligned} L^p &\geq \inf_n [n^p - (n-1)^p] \frac{t_n}{v_n} \geq \inf \frac{n^{p-1}}{v_n} t_n = \inf_n n^{2p+\alpha-1} \sum_{i=1}^{\infty} w_i (h_{i,n} - h_{i+1,n})^p \\ &= \inf_n n^{2p-1} \sum_{i=1}^{\infty} \frac{1}{\left(\frac{i}{n}\right)^\alpha (i+n)^p (i+n+1)^p}. \end{aligned}$$

Now let $E_k = \{i \in \mathbb{N} : (k-1)n \leq i \leq kn\}$ for $k = 1, 2 \cdots$. For $i \in E_k$, we have

$$\left(\frac{i}{n}\right)^\alpha (i+n)^p (i+n+1)^p \leq k^\alpha n^{2p} (k+1)^p (k+2)^p,$$

so

$$\sum_{i \in E_k} \frac{1}{\left(\frac{i}{n}\right)^\alpha (i+n)^p (i+n+1)^p} \geq \frac{n}{k^\alpha n^{2p} (k+1)^p (k+2)^p}.$$

Hence

$$L^p \geq \sum_{k=1}^{\infty} \frac{1}{k^\alpha (k+1)^p (k+2)^p}.$$

Since $\|e_1\|_{p,v} = 1$ and

$$\|He_1\|_{p,w,\Delta}^p = \sum_{n=1}^{\infty} \frac{1}{n^\alpha (n+1)^p (n+2)^p},$$

and also $L = \inf_{x \in l_p(v)} \frac{\|Hx\|_{p,w,\Delta}}{\|x\|_{p,v}} \leq \frac{\|He_1\|_{p,w,\Delta}}{\|e_1\|_{p,v}}$, which concludes the proof. □

REFERENCES

[1] *G. Bennett*, Factorizing the classical inequalities, Mem. Amer. Math. Soc., **576**(1996).

[2] *G. Bennett*, Lower bounds for matrices, Linear Algebra Appl., **82**(1986), 81-98.

[3] *G. Bennett*, Lower bounds for matrices II, Canad. J. Math., **44**(1992), 54-74.

[4] *J. W. Brown and R. V. Churchill*, Complex Variables and Applications, 8th edition, McGraw-Hill, 2009.

[5] *D. Foroutannia*, Upper bound and lower bound for matrix operators on weighted sequence spaces, Doctoral dissertation, Zahedan, 2007.

[6] *G. J. O. Jameson*, Norms and lower bounds of operators on the Lorentz sequence spaces d(w,1), Illinois J. Math., **43**(1999), 79-99.

[7] *G. J. O. Jameson and R. Lashkaripour*, Lower bounds of operators on weighted l_p spaces and Lorentz sequence spaces, Glasgow Math. J., **42**(2000), 211-223.

[8] *G. J. O. Jameson and R. Lashkaripour*, Norms of certain operators on weighted l_p spaces and Lorentz sequence spaces, J. Inequal. Pure Appl. Math., **3**(2002), No. 1, Article 6.

[9] *H. Kizmaz*, On certain sequence spaces I, Canad. Math. Bull., **25**(1981), No. 2, 169-176.

[10] *R. Lashkaripour and J. Fathi*, Norms of matrix operators on bv_p, J. Math. Inequal., **6**(2012), No. 4, 589-592.

[11] *R. Lyons*, A lower bound for the Cesaro operator, Proc. Amer. Math. Soc., **86**(1982), 694.