FURTHER RESULTS ON DISTANCE-BALANCED GRAPHS

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Distance-balanced graphs are graphs in which for every edge $e = uv$ the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. In this paper, we study this property under some graph operations. Also, we obtain lower and upper bounds on some topological indices of distance-balanced graphs.

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1. Introduction

The distance $d(u, v)$ between the vertices u and v of a graph G is equal to the length of a shortest path that connects u and v. For an edge $e = ab$ of G, let $n_a^G(e)$ be the number of vertices closer to a than to b. In other words, $n_a^G(e) = |\{u \in V(G)|d(u, a) < d(u, b)\}|$. In addition, let $n_0^G(e)$ be the number of vertices with equal distances to a and b, i. e., $n_0^G(ab) = |\{u \in V(G)|d(u, a) = d(u, b)\}|$. A graph G is said to be distance-balanced, if $n_a^G(e) = n_b^G(e)$, for each edge $e = ab \in E(G)$, see [1, 7, 17] for details. These graphs first studied by Handa [6] who considered distance-balanced partial cubes. In [9], Jerebič, Klavžar and Rall studied distance-balanced graphs in the framework of various kinds of graph products.

The Wiener index, W, is the first distance-based graph invariant to be used in chemistry [18]. For a graph G, it is equal to the count of all shortest distances in G. In other words, $W(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} d(u, v)$. Suppose $f = ab$ and $g = uv$ are arbitrary edges of G. Define $d_e(u, ab) = Min\{d(u, a), d(u, b)\}$ and $D(f, g) = Min\{d_e(u, f), d_e(v, f)\} = Min\{d_e(b, g), d_e(a, g)\}$. The edge Wiener index of a graph G is given by $W_e(G) = \frac{1}{2}\sum_{\{e,f\}\subseteq E(G)} D(e, f)$, see [11, 21] for details.

Following Yan et al. [19], the graph $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

The disjunction $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that (u_1, v_1) is adjacent to (u_2, v_2) whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$, see [10]. A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree k is called a $k-$regular graph

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or regular graph of degree k. The eccentricity of a vertex v is the greatest geodesic distance between v and any other vertex. The diameter of a graph is the maximum eccentricity of any vertex in the graph. The diameter of the graph G is denoted by $diam(G)$. A graph G is called nontrivial if $|V(G)| > 1$. Our other notations are standard and taken mainly form [3, 8, 15, 16, 20].

2. Main Results

All graphs considered here are finite and simple. In this section, we study the conditions under which some graph operations produce a distance-balanced graph.

Proposition 2.1. *Let G be a nontrivial connected graph. Then $R(G)$ is distance-balanced if and only if G is a path with $|V(G)| = 2$.*

Proof. Let G be a path with $|V(G)| = 2$. Then it is clear that $R(G)$ is distance-balanced. Conversely, we assume that $R(G)$ is a distance-balanced graph, where $|V(G)| > 2$. Then, there exists an edge $e = uv$ of G such that $deg_G(u) > 1$ or $deg_G(v) > 1$. Without loss of generality, we my assume that u is the end vertex of e with $deg_G(u) > 1$. Also, we assume that x is a new vertex corresponding to edge e of G. Then, $n_x^{R(G)}(xu) = 1$ and $n_u^{R(G)}(xu) > 1$. Thus $n_x^{R(G)}(xu) \neq n_u^{R(G)}(xu)$. Therefore $R(G)$, $|V(G)| > 2$, is not a distance-balanced graph and hence G is a path with $|V(G)| = 2$. $\square$

In what follows, $t(e)$, $e \in E(G)$, denotes the number of triangles containing edge e.

Proposition 2.2. *Let G and H be arbitrary, nontrivial and connected graphs. Then $G \vee H$ is distance-balanced if and only if G and H are regular graphs.*

Proof. We first assume that G and H are regular graphs. It is clear that, the diameter of $G \vee H$ is equal to 2. Therefore, for every edge $e = (a, x)(b, y) \in E(G \vee H)$, we have:

$$n_{(a,x)}^{G\vee H}(e) = deg_{G\vee H}((a, x)) - t(e), n_{(b,y)}^{G\vee H}(e) = deg_{G\vee H}((b, y)) - t(e).$$

On the other hand, it follows from the structure of $G \vee H$ that for each vertex $(a, b) \in V(G \vee H)$, $deg_{G\vee H}((a, b)) = |V(H)|deg_G(a) + |V(G)|deg_H(b) - deg_G(a)deg_H(b)$. Since G and H are regular graphs, for every $(a, x)(b, y) \in E(G\vee H)$, we have $n_{(a,x)}^{G\vee H}(e) = n_{(b,y)}^{G\vee H}(e)$ and thus $G \vee H$ is distance-balanced. Conversely, assume that $G \vee H$ is distance-balanced. It is clear that, for $x \in V(H)$ and every $ab \in E(G)$, $e = (a, x)(b, x) \in E(G \vee H)$. Since $G \vee H$ is distance-balanced this implies that $n_{(a,x)}^{G\vee H}(e) = n_{(b,y)}^{G\vee H}(e)$. On the other hand, it follows from the structure of $G \vee H$ that

$$\begin{aligned} n_{(a,x)}^{G\vee H}(e) &= deg_{G\vee H}((a, x)) - t(e) = |V(H)|deg_G(a) \\ &\quad + |V(G)|deg_H(x) - deg_G(a)deg_H(x) - t(e), \\ n_{(b,x)}^{G\vee H}(e) &= deg_{G\vee H}((b, x)) - t(e) = |V(H)|deg_G(b) \\ &\quad + |V(G)|deg_H(x) - deg_G(b)deg_H(x) - t(e). \end{aligned}$$

The two above equations imply $deg_G(a) = deg_G(b)$. Since G is connected this implies that G is r-regular for some r. In a similar way we can see that H is k-regular, for some k. □

Suppose G and H are graphs with disjoint vertex sets. Following Doslic [4], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y; z)$, is defined by identifying the vertices y and z in the union of G and H. Similarly, a link of G and H by vertices y and z is defined as the graph $(G \sim H)(y; z)$ obtained by joining y and z by an edge in the union of these graphs.

Proposition 2.3. *Suppose G and H are rooted graphs with respect to the rooted vertices of a and b, respectively. The graph $(G \cdot H)(a; b)$ is distance-balanced if and only if for each $e = uv \in E(G)$ and $f = xy \in E(H)$ the following conditions are satisfied:*

$$n_u^G(e) - n_v^G(e) = \begin{cases} |V(H)| - 1 & if\ d(v,a) < d(u,a) \\ 0 & if\ d(v,a) = d(u,a) \end{cases}, \quad (1)$$

$$n_x^H(f) - n_y^H(f) = \begin{cases} |V(G)| - 1 & if\ d(y,b) < d(x,b) \\ 0 & if\ d(y,b) = d(x,b) \end{cases}. \quad (2)$$

Proof. In the graph $(G \cdot H)(a; b)$, we put $r = a = b$. We partition edges of $(G \cdot H)(a; b)$ into the following two subsets:

$$A = \{e = uv \in E(G.H) | d(v, r) < d(u, r)\},$$
$$B = \{e = uv \in E(G.H) | d(v, r) = d(u, r)\}.$$

We first assume that $(G \cdot H)(a; b)$ is distance-balanced. Suppose $e = uv$ is an arbitrary edge of G. Then $e \in A$ or $e \in B$ and not both. If $e \in A$ then by our hypothesis, $n_u^{G \cdot H}(e) = n_v^{G \cdot H}(e)$. On the other hand, by definition of splice, $n_v^{G \cdot H}(e) = n_v^G(e) + |V(H)| - 1$ and $n_u^{G.H}(e) = n_u^G(e)$. Thus, $n_u^G(e) = n_v^G(e) + |V(H)| - 1$ and so $n_u^G(e) - n_v^G(e) = |V(H)| - 1$. Next we assume that $e \in B$. Again by our hypothesis, $n_u^{G \cdot H}(e) = n_v^{G \cdot H}(e)$ and by definition of splice we have, $n_v^{G \cdot H}(e) = n_v^G(e)$ and $n_u^{G.H}(e) = n_u^G(e)$. This implies that $n_u^G(e) = n_v^G(e)$. Therefore, the equation (1) is satisfied. In a similar way we can see that, for every edge e of H the equation (2) is satisfied.

Conversely, suppose that Eqs. (1,2) are satisfied and $e = uv \in A$ is arbitrary. Then $e \in E(G)$ or $e \in E(H)$ and not both. If $e \in E(G)$ then $n_u^{G.H}(e) = n_u^G(e)$ and $n_v^{G.H}(e) = n_v^G(e) + |V(H)| - 1$. This implies that $n_u^{G.H}(e) - n_v^{G.H}(e) = n_u^G(e) - (n_v^G(e) + |V(H)| - 1)$. Since $n_u^G(e) - n_v^G(e) = |V(H)| - 1$, $n_u^{G.H}(e) - n_v^{G.H}(e) = 0$, as desired. Suppose that $e \in E(H)$. Then $n_u^{G.H}(e) = n_u^H(e)$ and $n_v^{G.H}(e) = n_v^H(e) + |V(G)| - 1$, so $n_u^{G.H}(e) - n_v^{G.H}(e) = n_u^H(e) - (n_v^H(e) + |V(G)| - 1)$. But by the hypothesis, $n_u^H(e) - n_v^H(e) = |V(G)| - 1$, so $n_u^{G.H}(e) - n_v^{G.H}(e) = 0$. We now assume that $e \in B$ is arbitrary. If $e \in E(G)$ then by $n_u^{G.H}(e) = n_u^G(e)$ and $n_v^{G.H}(e) = n_v^G(e)$ we have $n_u^{G.H}(e) - n_v^{G.H}(e) = n_u^G(e) - n_v^G(e) = 0$. If $e \in E(H)$ then by $n_u^{G.H}(e) = n_u^H(e)$ and $n_v^{G.H}(e) = n_v^H(e)$ we have $n_u^{G.H}(e) - n_v^{G.H}(e) = n_u^H(e) - n_v^H(e) = 0$. Therefore, for every edge $e = uv \in B$, $n_u^{G.H}(e) = n_v^{G.H}(e)$ and for every edge $e = uv \in E(G \cdot H)$, $n_u^{G.H}(e) = n_v^{G.H}(e)$. This completes the proof. □

Corollary 2.1. *Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. Then*

$$(G_1 \cdot G_2 \cdot \cdots \cdot G_n)(r_1; r_2; \cdots; r_n)$$

is distance-balanced if and only if for each i, $1 \le i \le n$, and for each $e = uv \in E(G_i)$ the following system of equations are satisfied:

$$n_u^{G_i}(e) - n_v^{G_i}(e) = \begin{cases} \sum_{j=1, j\neq i}^{n} |V(G_j)| - (n-1) & if\ d(v, r_i) < d(u, r_i) \\ 0 & if\ d(v, r_i) = d(u, r_i) \end{cases}.$$

Proof. Induct on n. □

Proposition 2.4. *Suppose G and H are rooted graphs with respect to the rooted vertices of a and b, respectively. The graph $(G \sim H)(a; b)$ is distance-balanced if and only if $|V(G)| = |V(H)|$ and for each $e = uv \in E(G)$ and $f = xy \in E(H)$ the following conditions are satisfied:*

$$\begin{aligned} n_u^G(e) - n_v^G(e) &= \begin{cases} |V(H)| & if\ d(v, a) < d(u, a) \\ 0 & if\ d(v, a) = d(u, a) \end{cases}, \\ n_x^H(f) - n_y^H(f) &= \begin{cases} |V(G)| & if\ d(y, b) < d(x, b) \\ 0 & if\ d(y, b) = d(x, b) \end{cases}. \end{aligned}$$

Proof. The proof is similar to Proposition 2.3 and so omitted. □

Corollary 2.2. *Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. Then $(G_1 \sim G_2 \sim \cdots \sim G_n)(r_1; r_2; \cdots; r_n)$ is distance-balanced if and only if for each i, $1 \le i \le n$, $|V(G_i)| = |V(G_1)|$ and for each $e = uv \in E(G_i)$ the following system of equations are satisfied:*

$$n_u^{G_i}(e) - n_v^{G_i}(e) = \begin{cases} \sum_{j=1, j\neq i}^{n} |V(G_j)| & if\ d(v, r_i) < d(u, r_i) \\ 0 & if\ d(v, r_i) = d(u, r_i) \end{cases}.$$

Proof. Induct on n. □

We denote the complete graph and the cycle of order n by K_n and C_n, respectively. The complement or inverse of a graph G is a graph $\bar{G}$ on the same vertices such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

Proposition 2.5. *Let G be a distance-balanced graph and let e be an edge of $\bar{G}$. Then $G + e$ is not distance-balanced.*

Proof. Let G be a distance-balanced graph and let e be an edge of $\bar{G}$. Set $H = G+e$. Suppose H is distance-balanced graph. Then by Proposition 3.1 of [9], $H - e$ is not distance-balanced, which is a contradiction with the fact that $G = H - e$ is distance-balanced. □

We now obtain lower and upper bounds for distance-balanced graphs under some graph invariants. The Narumi-Katayama index was the first graph invariant defined by the product of some graph theoretical quantities applicable in chemistry. The Narumi-Katayama index of a graph G is given by $NK(G) = \prod_{v \in V(G)} deg(v)$, [5, 13].

Proposition 2.6. *Let G be a connected distance-balanced graph with $n > 2$ vertices. Then*

$$2^n \leqslant NK(G) \leqslant (n-1)^n,$$

where the left equality holds if and only if $G \cong C_n$ and the right equality holds if and only if $G \cong K_n$.

Proof. Since G is a connected distance-balanced graph with $n > 2$ vertices. Then for every vertex $v \in V(G)$, we have $2 \leqslant deg(G) \leqslant n-1$. Thus $2^n \leqslant NK(G) \leqslant (n-1)^n$. □

Proposition 2.7. *Let G be a connected distance-balanced graph with $n > 2$ vertices and $G \ncong K_n, C_n$. Then*

$$2^{n-2} \times 3^2 \leqslant NK(G) \leqslant (n-2)^n,$$

where the right equality holds if and only if G is a $(n-2)$-regular graph.

Proof. The left inequality is clear. On the other hand, there is not a connected distance-balanced graph with n vertices that has k vertices of degree $(n-1)$, $(0 < k < n)$. Therefore, if G is a connected distance-balanced graph and $G \ncong K_n$, then for each $v \in V(G)$, $deg(v) \leqslant (n-2)$. Also note that, every $(n-2)$-regular graph is a connected distance-balanced graph and this completes the proof. □

Proposition 2.8. *Let G be a connected distance-balanced graph with n vertices. Then*

$$\frac{n(n-1)}{2} \leqslant W(G) < [\frac{n}{2}]\Big(\binom{n}{2} - |E(G)|\Big) + |E(G)|,$$

where the left equality holds if and only if $G \cong K_n$.

Proof. It is clear that $W(K_n) \leqslant W(G)$. Let G be a connected distance-balanced graph with n vertices. If, there are two vertices a and b of G such that $d(a,b) = [\frac{n}{2}]+1$ and $a = a_0, a_1, a_2, ..., a_{[\frac{n}{2}]}, a_{[\frac{n}{2}]+1} = b$ is the shortest path connecting a and b, then, for the edge aa_1 of G, we have $n_{a_1}(aa_1) \geqslant [\frac{n}{2}] + 1$. Therefore, $n_a(aa_1) < n_{a_1}(aa_1)$ which is contradict by the fact that G is distance-balanced. Thus, $diam(G) \leqslant [\frac{n}{2}]$. This completes the proof. □

Proposition 2.9. *Let G be a connected distance-balanced graph with n vertices and $G \ncong K_n$. Then $\frac{n^2}{2} \leqslant W(G)$ with equality if and only if G is a $(n-2)$-regular graph.*

Proof. The proof is similar to Proposition 2.7 and so it is omitted. □

Our calculations on graphs with a small number of vertices suggest the following conjecture:

Conjecture 2.1. *If G be a connected distance-balanced graph with $n > 2$ vertices. Then*

$$W(G) \leqslant W(C_n).$$

Suppose G is a graph. The first Zagreb index of G is defined as $M_1(G) = \sum_{v\in V(G)} deg^2(v)$ and the second Zagreb of G is given by

$$M_2(G) = \sum_{uv\in E(G)} deg(u)deg(v),$$

see for details [2, 12, 14].

Proposition 2.10. *Let G be a connected distance-balanced graph with $n > 2$ vertices. Then*

$$4n \leqslant M_1(G) \leqslant n(n-1)^2,$$

and the lower and upper bounds are attained if and only if $G \cong C_n$ or $G \cong K_n$, respectively.

Proof. Since G is a connected distance-balanced graph with $n > 2$ vertices, for every vertex $v \in V(G)$, we have $2 \leqslant deg(G) \leqslant n-1$. Summing over all vertices, we get

$$4|V(G)| \leqslant \sum_{u\in V(G)} deg^2(u) \leqslant n(n-1)^2,$$

which proves the result. □

Proposition 2.11. *Let G be a connected distance-balanced graph with $n > 2$ vertices and $G \ncong K_n, C_n$. Then*

$$18 + 4(n-2) \leqslant M_1(G) \leqslant n(n-2)^2,$$

where the right equality holds if and only if G is a $(n-2)$-regular graph.

Proof. The proof is similar to Proposition 2.7 and so omitted. □

Proposition 2.12. *Let G be a connected distance-balanced graph with $n > 2$ vertices. Then*

$$4n \leqslant M_2(G) \leqslant \frac{n(n-1)^3}{2},$$

and the lower bound is attained if and only if $G \cong C_n$, for some n. Moreover, the upper bound is attained if and only if $G \cong K_n$.

Proof. Since G is a connected distance-balanced graph with $n > 2$ vertices, for every vertex $v \in V(G)$, we have $2 \leqslant deg(G) \leqslant n-1$ and $|V(G)| \leqslant |E(G)| \leqslant \binom{n}{2}$. Summing over all edges, we get

$$4|V(G)| \leqslant \sum_{uv\in E(G)} deg(u)deg(v) \leqslant \frac{n(n-1)^3}{2},$$

which proves the result. □

Proposition 2.13. *Let G be a connected distance-balanced graph with n vertices and $G \ncong K_n$. Then $M_2(G) \leqslant \frac{n(n-2)^3}{2}$, with equality if and only if G is a $(n-2)$-regular graph.*

Proof. The proof is similar to the proof of Proposition 2.7. □

Proposition 2.14. *Let G be a connected distance-balanced graph with n vertices. Then*

$$W_e(G) \leqslant \frac{1}{2}[\frac{n}{2}]\Big(|E(G)|(|E(G)|+1) - M_1(G)\Big).$$

Proof. Suppose G is a connected distance-balanced graph with n vertices. Then $diam(G) \leqslant [\frac{n}{2}]$ and hence $max_{f,g\in E(G)}\{D(e,f)\} \leqslant [\frac{n}{2}]$. On the other hand, the number of edge-pairs which have zero distance is equal to $\sum_{i=1}^{n}\binom{deg(v_i)}{2}$ and this completes the proof. □

Corollary 2.3. *Let G be a connected distance-balanced graph with n vertices. Then*

$$W_e(G) \leqslant \frac{1}{2}[\frac{n}{2}]\Big(|E(G)|(|E(G)|+1) - 4n\Big).$$

Proof. The proof follows from Propositions 2.10 and 2.14. □

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