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ON A CUBICAL SUBDIVISION OF THE SIMPLICIAL COMPLEX

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For a simplicial complex Δ *we study a particular case of the subdivision* Δ^{sub} *of* Δ *defined in* [3]. *We find the transformation maps sending the* f*- and* h*- vectors of* Δ *to the* f*- and* h*- vectors of* Δ^{sub} *along with some properties of the corresponding transformation matrices.*

Keywords: Simplicial complex, f- and h-vectors, barycentric subdivision of a cube.

1. Introduction

Motivated from [2] and [1], this article is about the study of the barycentric subdivision Δ^{sub} of the cubical complex Δ^c associated to a simplicial compelx Δ. A cubical complex is a union of unit cubes i.e. points, line segments, squares, cubes, and their n-dimensional counterparts [5]. They are used analogously to simplicial complexes and CW complexes in the computation of the homology of topological spaces.

A simplicial complex Δ on the ground set $[n]=\{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ such that if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$. An element F of Δ is called a *face* and inclusion wise maximal faces are called *facets*. The dimension of a face F is defined by $\dim(F) = |F| - 1$, where $|F|$ is the cardinality of F. The *dimension* of a simplicial complex Δ is defined as

$$\dim \Delta = \max\{\dim(F) : F \in \Delta\}.$$

Let f_k be the number of k-dimensional faces of Δ. We set $f_{-1} = 1$ corresponding to the empty set $\phi \in \Delta$. For a $(d-1)$-dimensional simplicial complex Δ, the vector $(f_{-1}, f_0, f_1, \dots, f_{d-1})$ is called the f-vector of Δ. The h-vector $(h_0, h_1, \dots, h_d)$ of Δ is defined by the relations

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

The h-vector of Δ plays an important role in studying the algebraic properties of the Stanley Resiner ideal R/I_Δ associated to Δ. Here $R = k[x_1, \dots, x_n]$ is the polynomial ring in n variables over the field k and I_Δ is defined as

$$I_\Delta = (x_{i_1} \cdots x_{i_r} | \{i_1, \dots, i_r\} \notin \Delta).$$

For more details about algebraic applications we refer the reader to [4].

A n-dimensional polytope which is the convex hull of the $n + 1$ vertices is called a n-simplex. For example a 3-simplex is a tetrahedron. We denote a n-simplex by σ_n. Each k-dimensional face of a simplicial complex Δ is a k-simplex. Let $\mathcal{F}(\Delta) = \{F_1, \dots, F_t\}$ be set

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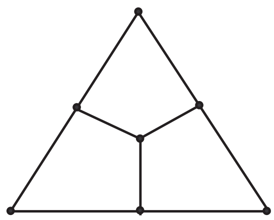

Fig. 1. 2-dimensional cubi-cal complex

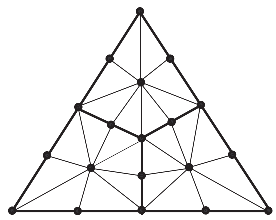

Fig. 2. 2-dimensional sub-divided complex

of facets of Δ. Then we write $\Delta =< F_1, \ldots, F_t >$ and say Δ is generated by $\mathcal{F}(\Delta)$. Thus to define a subdivision of a simplicial complex Δ, it is enough to consider the subdivision of its facets.

The subdivision Δ^{sub} studied in this article is a particular case of the subdivision defined in [3]. We define the subdivision Δ^{sub} of a simplicial complex Δ by defining subdivisions of the facets of Δ generically realized as standard simplices. Let

$$\sigma_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_i \geq 0 \text{ for all } i \text{ and } x_0 + \cdots + x_n = 1\}$$

be the geometric realization of a standard n-simplex. For $j = 0, \ldots, n$ we define

$$C_j = \{(x_0, \ldots, x_n) \in \sigma_n | x_j \geq x_i \text{ for all } i\}.$$

Clearly, C_j is a polytope. For $i = 0, \ldots, n$ and $i \neq j$, its facets are given by $\{(x_0, \ldots, x_n) \in \sigma_n \mid x_i = 0\}$ and $\{(x_0, \ldots, x_n) \in \sigma_n \mid x_j \geq x_i\}$. The vertices of C_j are $\frac{1}{|A|} \sum_{i \in A} e_i$ for subsets $A \subseteq \{0, \ldots, n\}$ where $j \in A$. This identifies C_j as a polytope combinatorially isomorphic to an n-dimensional cube. For $B \subseteq \{0, \ldots, n\}$ we have that $\bigcap_{j \in B} C_j$ is the face of the C_j, which is given by setting the coordinates in B equal. Thus the C_j are cubical complex subdividing the simplex σ_n (see Figure 1).

Now we form the barycentric subdivision of this cubical complex. This defines a simplicial complex subdividing σ_n. One checks that this procedure applied to all facets of Δ is compatible and defines a simplicial subdivision Δ^{sub} of Δ (see Figure 2). Note that the cubical complex Δ^c is not a simplicial complex rather it is collection of hypercubes the way we have simplicies in a simplicial complex. Thus an i-dimensional face F^c of Δ^c is an i-cube with dimension $\dim(F^c) = |F^c| - 1$. The dimension of Δ^c is defined to be $\max\{\dim(F^c) \mid F^c \in \Delta^c\}$. We denote by f_i^c the number of i-dimensional faces in Δ^c. For example in Figure 1, $f_0^c = 7$, $f_1^c = 9$ and $f_2^c = 3$.

We organize this manuscript as follows. The second section contains results related to the f- and h-vectors transformations. Proposition 1 counts number of i-dimensional faces of the cubical complex Δ^c while Theorem 1 provides relations to compute f_j^{sub} in term of f_k^c. Corollary 1 is the f-vector transformation sending the f-vector of Δ to the f-vector of Δ^{sub}. Proposition 2 deals with the h-vector transformation. In Section **3**, we study some properties of the transformation matrices obtained from transformation maps of Section **2**. Proposition 2 states that the transformation matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{H}_{d-1}$ are similar and diagonizable. Proposition 3 gives some information about the eigen vectors of $\mathfrak{F}_{d-1}$. The main result of this section, Theorem 2, gives a nice formula to compute determinant of $\mathfrak{F}_{d-1}$.

2. The f- and h- vectors transformations

First we define some terminologies. Let σ_i be an i-simplex and σ_i^c be its cubical simplex consisting of $i+1$ number of i-cubes. These i-cubes share a common vertex. We call this common vertex as inner vertex. Any face (of cube) of σ_i^c which contains the inner vertex is called an inner face (of cube). Any other face (of cube) of σ_i^c lies on the boundary of σ_i^c.

Proposition 1. *Let Δ be a $(d-1)$-dimensional simplicial complex and Δ^c be the cubical complex. Then for $0 \leq i \leq d-1$, the number of i-dimensional faces of Δ^c is given by $f_i^c = \sum_{j=i}^{d-1} \binom{j+1}{i} f_j$, where $f = (f_0, \ldots, f_{d-1})$ be the f-vector of the simplicial complex Δ.*

Proof. Firstly, note that each i-dimensional face F_i^c of Δ^c can be obtained from a face F_j of dimension j of Δ for $i \leq j \leq d-1$. This fix the range of j in the required formula.

Secondly, it enough to consider a standard j-simplex σ_j. We are interested to calculate the number of k-dimensional cubes which lie inside the subdivided cubical j-simplex σ_j^c for $0 \leq k \leq j$. By the geometric definition, the inner vertex share an edge with each inner vertex of $(j-1)$-dimensional simplicies lies the boundary of σ_j^c. Since there are j+1 such simplies we count these number of edges as $\binom{j+1}{1}$. Now each pair of inner edges of σ_j^c contribute to an inner 2-cube, hence number of 2-cubes is $\binom{j+1}{2}$ and so on. Finally, each combination of j inner edges contribute to a j-dimensional inner cube and hence, the number of such j-cubes is given by $\binom{j+1}{j}$.

Since these calculations take into account only the inner cubes of Δ^c, if we consider each j-dimensional face of Δ as an j-simplex, it follows that the total numbers of i-faces of Δ^c is given by $\binom{i+1}{i} f_i + \binom{i+2}{i} f_{i+1} + \ldots + \binom{d}{i} f_{d-1} = \sum_{j=i}^{d-1} \binom{j+1}{i} f_j$. □

To prove the remaining results of this section we need following combinatorial result.

Lemma 1. *Let $\mathcal{C}(n,i)$ be the number of i-dimensional faces of a n-cube. Then*

$$\sum_{n_{i-1}=i}^{n_i-1} \{\cdots\{\sum_{n_0=1}^{n_1-1}\{\sum_{j_0=0}^{n_0-1} \mathcal{C}(n_0,j_0)\}\mathcal{C}(n_1,n_0)\}\cdots\}\mathcal{C}(n_i,n_{i-1}) = \mathcal{H}(n,i), \qquad (\mathbf{2}.1)$$

where $n_i = n$ and

$$\mathcal{H}(n,i) = \sum_{j=0}^{i+1} (-1)^j \binom{i+1}{j} (2(i-j)+3)^n. \qquad (\mathbf{2}.2)$$

Proof. We prove it by using induction on i. It is well known that $\mathcal{C}(n,i) = 2^{n-i}\binom{n}{i}$. For $i=0$, we have $n_0 = n$ and $\sum_{j_0=0}^{n-1} \mathcal{C}(n,j_0) = \sum_{j_0=0}^{n-1} 2^{n-j_0}\binom{n}{j_0} = (2+1)^n - 1 = 3^n - 1 = \mathcal{H}(n,0)$. Suppose Equation (1) is true for i. Now, for $i+1$ we have that $n_{i+1} = n$ and

$$\sum_{n_i=i+1}^{(n=n_{i+1})-1} \{\ldots\{\sum_{n_0=1}^{n_1-1}\{\sum_{j_0=0}^{n_0-1} \mathcal{C}(n_0,j_0)\}\mathcal{C}(n_1,n_0)\}\ldots\}\mathcal{C}(n_{i+1},n_i)$$

$$= \sum_{n_i=i+1}^{n-1}\{\sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}(2(i-j)+3)^{n_i}\}\mathcal{C}(n,n_i)$$

$$= \sum_{n_i=i+1}^{n-1}\{\sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}(2(i-j)+3)^{n_i}\}2^{n-n_i}\binom{n}{n_i}.$$

Using binomial theorem and simplification we get

$$= \sum_{j=0}^{i+2}(-1)^j\binom{i+2}{j}(2(i-j+1)+3)^n.$$

Hence Equation (1) is true by mathematical induction. □

We use factor $\mathcal{H}(n,i)$ defined in above lemma in the remaining part of this section. Note that $\mathcal{H}(n,i) < \mathcal{H}(n+1,i)$.

Lemma 2. *Let Δ be a $(d-1)$-dimensional simplicial complex. For $0 \le j \le d-1$, the number of j-dimensional faces of the subdivided simplicial complex Δ^{sub} is given by*

$$f_j^{\mathrm{sub}} = \sum_{k=j}^{d-1} \mathcal{H}(k, j-1) f_k^c,$$

where f_k^c be the number of k-dimensional faces of the cubical complex Δ^c.

Proof. By definition, the barycentric subdivision of Δ^c is a simplicial complex Δ^{sub} on the ground set $\Delta^c \setminus \{\emptyset\}$. The j-dimensional faces of Δ^{sub} are the strictly increasing chains $F_0^c \subset F_1^c \subset \cdots \subset F_j^c$ (of length j) of faces in $\Delta^c \setminus \{\emptyset\}$. We fix j and some k-dimensional face F_j^c and count the chains of length j whose top element is F_j^c. If $j = 0$ then by definition

$$f_0^{sub} = \sum_{k=0}^{d-1} f_j^c = \sum_{k=0}^{d-1} \mathcal{H}(k,-1)\, f_j^c.$$

Assume $j > 0$. The dimensions k_0 of F_0^c, …, k_{j-1} of F_{j-1}^c are a strictly increasing sequence of numbers $0 \le k_0 < \cdots < k_{j-1} < k \le d-1$. Fixing these numbers there $\mathcal{C}(k, k_{j-1})$ choices for F_{j-1}^c, $\mathcal{C}(k_{j-1}, k_{j-2})$ choices for F_{j-2}^c, …, $\mathcal{C}(k_1, k_0)$ choices for F_0^c. Summing up over the choices we get

$$\sum_{k_{j-1}=j-1}^{k-1} \cdots \sum_{k_1=1}^{k_2-1} \sum_{k_0=0}^{k_1-1} \mathcal{C}(k_1,k_0)\mathcal{C}(k_2,k_1)\cdots\mathcal{C}(k_j,k_{j-1}) = \mathcal{H}(k_j, j-1).$$

Now there are f_j^c choices for F_j^c and its dimension k must be at least j. This yields $f_j^{sub} = \sum_{k=j}^{d-1} \mathcal{H}(k,j-1) f_k^c$. □

On combining the results of Proposition 1 and Lemma 2, we get the following corollary.

Theorem 1. *Let Δ be a $(d-1)$-dimensional simplicial complex and Δ^{sub} be the subdivided simplicial complex. The number of i-dimensional faces of Δ^{sub} is given by $f_i^{\mathrm{sub}} = \sum_{j=i}^{d-1} \sum_{k=j}^{d-1} \mathcal{H}(j,i-1)\binom{k+1}{j} f_k$, where f_j is the number of j-dimensional faces of Δ.*

Note that $f_i < f_i^{\mathrm{sub}}$, for any $0 \le i \le d-1$. Now extending these results to h-vector transformation, we give the following proposition.

Corollary 1. *Let Δ be a $(d-1)$-dimensional simplicial complex and Δ^{sub} be the subdivided simiplicial complex. The h-vector of Δ^{sub} is given by $h_i^{\mathrm{sub}} =$*

$$(-1)^i \binom{d}{i} h_0 + \sum_{j=1}^{i} \sum_{k=j-1}^{d-1} \sum_{l=k}^{d-1} \sum_{m=0}^{l+1} (-1)^{i-j} \binom{d-j}{i-j} \binom{l+1}{k} \binom{d-m}{d-l-1} \mathcal{H}(k, j-2) h_m,$$

where $h = (h_0, h_1, \ldots, h_{d-1})$ is the h-vector of Δ.

Proof. From the definition of the h-vector of Δ , we have

$$h_i^{\mathrm{sub}} = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}^{\mathrm{sub}} = (-1)^i \binom{d}{i} f_{-1}^{\mathrm{sub}} + \sum_{j=1}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}^{\mathrm{sub}}.$$

As $f_{-1}^{\mathrm{sub}} = h_0^{\mathrm{sub}} = h_0 = 1$, we have

$$h_i^{\mathrm{sub}} = (-1)^i \binom{d}{i} h_o + \sum_{j=1}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}^{\mathrm{sub}} \quad (\mathbf{2}.3)$$

If we substitute the expression of f_{j-1}^{sub} from Lemma 2 in Equation **2**.3, we get

$$h_i^{\text{sub}} = (-1)^i \binom{d}{i} h_0 + \sum_{j=1}^{i} \sum_{k=j-1}^{d-1} (-1)^{i-j} \binom{d-j}{i-j} \mathcal{H}(k, j-2) f_k^c. \qquad (\mathbf{2}.4)$$

Using the expression of f_k^c given in Proposition 1 in Equation **2**.4, we have

$$h_i^{\text{sub}} = (-1)^i \binom{d}{i} h_0 + \sum_{j=1}^{i} \sum_{k=j-1}^{d-1} \sum_{l=k}^{d-1} (-1)^{i-j} \binom{d-j}{i-j} \binom{l+1}{k} \mathcal{H}(k, j-2) f_l. \qquad (\mathbf{2}.5)$$

Since for any $0 \leq k \leq d-1$, $f_{k-1} = \sum_{i=0}^{k} \binom{d-i}{d-k} h_i$, so by Equation **2**.5 we have the required result. □

3. Transformation Matrices

For a $(d-1)$-dimensional simplicial complex Δ we denote by $\mathfrak{F}_{d-1} = (f_{ij})_{0 \leq i,j \leq d} \in \mathbb{R}^{(d+1)\times(d+1)}$ and $\mathfrak{H}_{d-1} = (h_{ij})_{0 \leq i,j \leq d} \in \mathbb{R}^{(d+1)\times(d+1)}$ the matrices of transformations that send f- and h-vectors of Δ to f- and h-vectors of Δ^{sub}, respectively. Thus $f_{i-1}^{\text{sub}} = \sum_{j=0}^{d} f_{ij} f_{j-1}$ and $h_i^{\text{sub}} = \sum_{j=0}^{d} h_{ij} h_j$, where $0 \leq i \leq d$. Note that entries of the matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{H}_{d-1}$ can be computed from Theorem 1 and Corollary 1. For example for $d = 3$ we obtained following matrices.

$$\mathfrak{F}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 30 \\ 0 & 0 & 0 & 24 \end{pmatrix}, \mathfrak{H}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 16 & 14 & 10 & 7 \\ 7 & 10 & 14 & 16 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 2. *For a $(d-1)$-dimensional simplicial complex Δ:*

(a) The matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{H}_{d-1}$ are similar.

(b) The matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{H}_{d-1}$ are diagonizable and have the eigenvalue 1 of multiplicity 2 and eigenvalues $\lambda_k = (k+1)!2^k$ of multiplicity 1 for each $k = 1, \ldots, d-1$.

Proof. (a) Since the transformation sending f-vector of Δ to h-vector of Δ is an invertible linear transformation, thus by Theorem 1 and Corollary 1 the matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{H}_{d-1}$ are similar.

(b) Consider $\mathfrak{F}_{d-1}$. Clearly $\mathfrak{F}_{d-1}$ is an upper triangular matrix with diagonal entries $1, 1, 4, 24, \ldots, d!2^{d-1}$. Since the matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{H}_{d-1}$ are similar, thus the result follows.

□

Proposition 3. *Let $d \geq 3$ and $r_1(d), r_1^*(d), r_2(d), \ldots, r_d(d)$ be some eigenvectors of the matrix $\mathfrak{F}_{d-1}$, where $r_1(d), r_1^*(d)$ are eigenvectors for the eigenvalue 1 and $r_k(d)$ is an eigenvector for the eigenvalue $k!2^{k-1}, 2 \leq k \leq d$. Then the vectors $r_1(d+1) = (r_1(d), 0), r_1^*(d+1) = (r_1^*(d), 0)$ and $r_k(d+1) = (r_k(d), 0)$ are eigenvectors of $\mathfrak{F}_d$ for the eigenvalues $1, 1, 4, 24, \ldots, d!2^{d-1}$.*

Proof. Since by Theorem 1, for a $(d-1)$-dimensional simplicial complexes Δ and its subdivided simplicial complex Δ^{sub}, we have $f_i^{\text{sub}} = \sum_{j=i}^{d-1} \sum_{k=j}^{d-1} \mathcal{H}(j, i-1) \binom{k+1}{j} f_k$. Thus clearly $\mathfrak{F}_{d-1}$ and $\mathfrak{F}_d$ are upper triangular matrices. Moreover, coefficients f_{ij} of f_{j-1} are same for $0 \leq i \leq d$, $0 \leq j \leq d$ in both matrices $\mathfrak{F}_{d-1}$ and $\mathfrak{F}_d$. Thus

$$\mathfrak{F}_d = \left(\begin{array}{ccc|c} & & & f_{1(d+1)} \\ & \mathfrak{F}_{d-1} & & \vdots \\ & & & f_{d(d+1)} \\ \hline 0 & \ldots & 0 & f_{(d+1)(d+1)} \end{array}\right)$$

Hence the result follows.

□

Theorem 2. *Let $\mathfrak{F}_{d-1}$ be the matrix of transformation sending the f-vector of Δ to the f-vector of Δ^{sub}. Then determinant of $\mathfrak{F}_{d-1}$ is $\mid \mathfrak{F}_{d-1} \mid = \prod_{i=0}^{d-1}(i+1)!2^i$.*

Proof. The number of k-dimensional faces of the subdivided simplicial complex Δ^{sub} is calculated using l dimensional faces of the given simplicial complex Δ, where $k \leq l \leq d-1$. So $\mathfrak{F}_{d-1}$ is an upper triangular matrix and hence its determinant $\mid \mathfrak{F}_{d-1} \mid$ is given by the product of its diagonal entries. By Theorem 1, the diagonal entries of $\mathfrak{F}_{d-1}$ are given by $(i+1)\mathcal{H}(i,i-1)$ for $0 \leq i \leq d-1$. From Equation **2**.2, $\mathcal{H}(i,i-1) = 2^i \sum_{j=0}^{i}(-1)^j \binom{i}{j}(i-j+\frac{1}{2})^i$. Thus it remains to prove $\sum_{j=0}^{i}(-1)^j \binom{i}{j}(i-j+\frac{1}{2})^i = i!$. For this we proceed as follows. The ith derivative of a function $f(x)$ is defined by

$$f^{(i)}(x) = \lim_{h\to 0} \frac{\Delta_h^i f(x)}{h^i}, \qquad (\mathbf{3}.1)$$

where $\Delta_h^i f(x) = \sum_{j=0}^{i}(-1)^j \binom{i}{j} f(x+(i-j)h)$ is the ith forward difference of $f(x)$. Now applying Equation **3**.1 on $f(x) = x^i$, we get

$$f^{(i)}(x) = i! = \lim_{h\to 0} \frac{\sum_{j=0}^{i}(-1)^j \binom{i}{j}(x+(i-j)h)^i}{h^i}. \qquad (\mathbf{3}.2)$$

Obviously Equation **3**.2 is true for any value of x. In particular for $x = \frac{1}{2}h$, we have

$$i! = \lim_{h\to 0} \frac{\sum_{j=0}^{i}(-1)^j \binom{i}{j}(\frac{1}{2}h+(i-j)h)^i}{h^i} = \sum_{j=0}^{i}(-1)^j \binom{i}{j}(\frac{1}{2}+i-j)^i,$$

as required. □

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