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TWO-CATALAN NUMBERS: COMBINATORIAL INTERPRETATION AND LOG-CONVEXITY

Yousra GHEMIT[1] and Moussa AHMIA[2]

We provide the two-Catalan triangle and in particular the two-Catalan numbers, with a recurrence relation and a combinatorial interpretation that leads us to prove the log-convexity of those numbers. Furthermore we demonstrate that the rows of two-Catalan triangle form a log-concave sequence.

Keywords: Catalan numbers, Tow-Catalan numbers, log-convexity, log-concavity.
MSC2010: 05A 19, 05A 20.

1. Introduction

The Catalan numbers are a sequence of natural integers which has the explicit formula

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n},$$

these numbers play an important role in mathematics. As good references, we refer the reader to [15].

The first values of Catalan numbers are given by A000108 in OEIS [19] are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, \dots$$

The Catalan numbers satisfy the following recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

And they have as generating function

$$C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Combinatorial interpretations: The Catalan numbers have a various combinatorial interpretations [12, 13, 18], we mention from them that of binary trees and Dyck paths:

[1]University of Mohamed Seddik Benyahia, LMAM laboratory, BP 98 Ouled Aissa Jijel, Algeria, e-mail: ygm98190@gmail.com

[2]University of Mohamed Seddik Benyahia, LMAM laboratory, BP 98 Ouled Aissa Jijel, Algeria, e-mail:moussa.ahmia@univ-jijel.dz

- The Catalan number C_n counts the number of rooted plane trees with n edge (for more details see [13]).
- The Catalan number C_n counts the number of Dyck Paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ and never falling below the x-axis, or similarly, the number of lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ and $(1,0)$, never rising above the line $y=x$.

The bi^{s}nomial coefficient is defined as the k-th coefficient in the development

$$(1+t+\cdots+t^s)^n=\sum_{k=0}^{sn}\binom{n}{k}_s t^k,$$

and satisfies the following

- Symmetry relation
$$\binom{n}{k}_s=\binom{n}{sn-k}_s \tag{1}$$
- Longitudinal recurrence relation (generalized Pascal formula).
$$\binom{n}{k}_s=\sum_{j=0}^{s}\binom{n-1}{k-j}_s. \tag{2}$$

See for instance [5, 4].

The first result dealing with unimodality of bi^{s}nomial coefficients is due to Belbachir and Szalay [7] who proved that any ray crossing Pascal's triangle provides a unimodal sequence. Then, Ahmia and Belbachir in [1, 2, 3] established respectively the strong log-convexity, the unimodality and log-concavity properties for the bi^{s}nomial coefficients.

For an odd integer s, Belbachir and Igueroufa [6] introduced the s-Catalan number as

$$C_n^{(s)}=\binom{2n}{sn}_s-\binom{2n}{sn+1}_s, \tag{3}$$

where $\binom{2n}{sn}_s$ is the central bi^{s}nomial coefficients. Linz [16] generalized the definition of s-Catalan numbers for all positive integers s, and he gave a combinatorial description for these numbers in terms of Littlewood-Richardson coefficients.

In this paper we consider the case of $s=2$ of the s-Catalan number defined by the relation (3) because the general case is complicated to study by the same techniques used in this paper. In Section 2, we define the coefficients of two-Catalan triangle in which we call the coefficients of the first column of this triangle by *two-Catalan numbers*, then we give a recurrence relation. In Section 3, we give a combinatorial interpretation of the coefficients of two-Catalan triangle by a subset of the set of vertically constrained Motzkin-like paths introduced by Irvine et al. [14]. Using this combinatorial interpretation, we prove in Section 4 the log-convexity of two-Catalan numbers. In Section 5, we establish that the rows of two-Catalan triangle form a log-concave sequence.

2. **Two-Catalan numbers**

First off all, we stat by the following definition.

Definition 2.1. *Let n and k be two positive integers. We define the coefficients of two-Catalan triangle as follows*

$$C_{n,k}^{(2)} := \binom{2n}{2n+k}_2 - \binom{2n}{2n+k+1}_2,$$

for $0 \le k \le 2n$.
In particular, we call the coefficients of two-Catalan triangle for $k = 0$ by "the two-Catalan numbers" denoted $C_n^{(2)}$,

$$C_n^{(2)} := C_{n,0}^{(2)} = \binom{2n}{2n}_2 - \binom{2n}{2n+1}_2,$$

where $\binom{2n}{2n}_2$ is the central trinomial coefficient.

This definition leads us to the following proposition.

Proposition 2.1. *The coefficients of the two-Catalan triangle satisfy*

$$C_{n+1,0}^{(2)} = C_{n,0}^{(2)} + C_{n,1}^{(2)} + C_{n,2}^{(2)}, \tag{4}$$

$$C_{n+1,1}^{(2)} = C_{n,0}^{(2)} + 3C_{n,1}^{(2)} + 2C_{n,2}^{(2)} + C_{n,3}^{(2)}, \tag{5}$$

$$C_{n+1,k}^{(2)} = C_{n,k-2}^{(2)} + 2C_{n,k-1}^{(2)} + 3C_{n,k}^{(2)} + 2C_{n,k+1}^{(2)} + C_{n,k+2}^{(2)}, \text{ for } k \ge 2, \tag{6}$$

where $C_{0,0}^{(2)} = 1$ and $C_{n,k}^{(2)} = 0$ unless $2n \ge k \ge 0$.

Proof. From Definition 2.1, and by applying the recurrence relation (2) twice in succession on left side of (6) we obtain, for $k \ge 2$,

$$\begin{aligned} C_{n+1,k}^{(2)} &= \binom{2n+2}{2n+k+2}_2 - \binom{2n}{2n+2+k+3}_2 = \binom{2n+1}{2n+k}_2 - \binom{2n+1}{2n+k+3}_2 \\ &= \binom{2n}{2n+k-2}_2 + \binom{2n}{2n+k-1}_2 + \binom{2n}{2n+k}_2 - \binom{2n}{2n+k+1}_2 - \binom{2n}{2n+k+2}_2 \\ &\quad - \binom{2n}{2n+k+3}_2 \qquad (7) \\ &= C_{n,k-2}^{(2)} + 2C_{n,k-1}^{(2)} + 3C_{n,k}^{(2)} + 2C_{n,k+1}^{(2)} + C_{n,k+2}^{(2)}. \end{aligned}$$

To prove (5) and (6), it suffices to use the symmetry property (1) on the right side of (7) on the term $\binom{2n}{2n+k-2}_2$ for $k = 1$, and on the terms $\binom{2n}{2n+k-2}_2$ and $\binom{2n}{2n+k-1}_2$ for $k = 0$, we obtain respectively:

$$\binom{2n}{2n}_2 + 2\binom{2n}{2n+1}_2 - \binom{2n}{2n+2}_2 - \binom{2n}{2n+3}_2 - \binom{2n}{2n+4}_2$$

and

$$\binom{2n}{2n}_2 - \binom{2n}{2n+3}_2,$$

which are after simplify the right side of $C^{(2)}_{n+1,1}$ and $C^{(2)}_{n+1,0}$ respectively. □

Table 2 gives us the first values of the two-Catalan triangle.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	**1**										
1	**1**	1	1								
2	**3**	6	6	3	1						
3	**15**	36	40	29	15	5	1				
4	**91**	232	280	238	154	76	28	7	1		
5	**603**	1585	2025	1890	1398	837	405	155	45	9	1

Table 2: Two-Catalan Triangle.

The two-catalan numbers equal also the Riordan numbers of even indices [8], which are given by the formula

$$r_n = \frac{1}{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k}\binom{n-k-1}{k-1}$$

They are related to the Catalan numbers by the relation [8]

$$r_n = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}C_j.$$

The Riordan numbers have many combinatorial interpretations, see [8]. And they have as generating function

$$R(x) = \sum_{n\geq 0} r_n x^n = \frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)}.$$

The first few Riordan numbers r_n are $\mathbf{1}, 0, \mathbf{1}, 1, \mathbf{3}, 6, \mathbf{15}, 36, \mathbf{91}, 232, \mathbf{603}$. See OEIS [19, A005043].

Then the two-catalan numbers satisfy the following identity:

$$C^{(2)}_n = \sum_{j=0}^{2n}(-1)^j\binom{2n}{j}C_j. \tag{8}$$

3. Combinatorial interpretation of Two-Catalan numbers

According to [8], we have the following combinatorial interpretation for the two-Catalan numbers.

Corollary 3.1. *The two-Catalan number $C^{(2)}_n$ counts the number of short bushes with $2n$ edges in which no vertex has outdegree one (i.e., each internal node has at least two edges).*

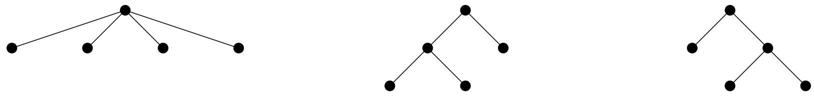


FIGURE 1. The short bushes for $n = 2$.

As an example, the Figure 1 shows the three short bushes for $n = 2$.

Irvine et al. [14] introduced the vertically constrained Motzkin-like paths as the lattice paths formed from the set of step vectors $\mathcal{A} = \{(1,0),(1,1),(1,-1),(0,1),(0,-1)\}$ (i.e., the allowed steps are: East step, North-East step, South-East step, North step and South step) with the constraint :"no consecutive vertical steps are allowed". Then, the authors distinguished the following four classes of vertically constrained lattice paths:

- $\mathbf{A^H}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ in the half-plane.
- $\mathbf{A^H_R}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ in the half-plane, in which the leading step is restricted to $\{(1,0),(1,1),(1,-1)\}$.
- $\mathbf{A^Q}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ restricted to the quarter-plane.
- $\mathbf{A^Q_R}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ restricted to the quarter-plane, in which the leading step is restricted to $\{(1,0),(1,1),(1,-1)\}$.

Then they gave the recurrence relations of each case.

Inspired by these paths, we define a subset of vertically constrained Motzkin-like paths, and in the same way we prove in Theorem 3.1 that our paths satisfy the recurrence relation of the coefficients of two-Catalan triangle.

Definition 3.1. *Let $\mathcal{A}^Q_{R,2}$ be the set of vertically constrained Motzkin-like paths from $(0,0)$ in the upper right quarter-plane (Q) in which the leading step is not a vertical step, satisfying the condition that in each point (i,j), if $j = 1$ no horizontal step is allowed on the horizontal level $y = 0$ (the x-axis) on the level just before this point, and if $j = 0$ no horizontal step is allowed on the horizontal levels $y = 0$ and $y = 1$ on the level just before this point. We denote by $\mathcal{A}^Q_{R,2}(n,k)$ the set of paths of type $\mathcal{A}^Q_{R,2}$ from $(0,0)$ to (n,k), and by $a^Q_{R,2}(n,k)$ the cardinality of $\mathcal{A}^Q_{R,2}(n,k)$, i.e., $a^Q_{R,2}(n,k)$ counts the number of paths in $\mathcal{A}^Q_{R,2}(n,k)$.*

For instance, the paths of $\mathcal{A}^Q_{R,2}(2,0)$ are shown in Figure 2.

Theorem 3.1. *Let n and k be two positive integers, then*

$$C^{(2)}_{n,k} = a^Q_{R,2}(n,k).$$

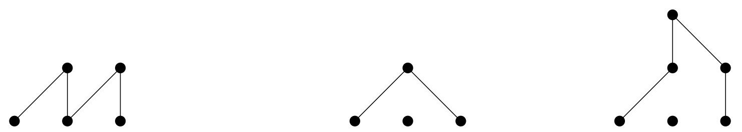


Figure 2. The paths of $\mathcal{A}^{Q}_{R,2}(2,0)$.

Proof. What we are going to do is to prove that $a^{Q}_{R,2}(n,k)$ satisfies the same recurrence relation of Proposition 2.1, and then conclude the equality $a^{Q}_{R,2}(n,k) = C^{(2)}_{n,k}$. For $k \geq 2$, there are five possible cases as it is illustrated in Figure 3. For $k = 1$ (resp. for $k = 0$) and as no horizontal step is allowed on the x-axis (resp. on the horizontal levels $y = 0$ and $y = 1$), we delete any horizontal step there, and of course any step extends below the x-axis because we should not forget that our paths are restricted to the upper right quarter-plane (Q) as it is illustrated in Figure 4 (resp. in Figure 5).

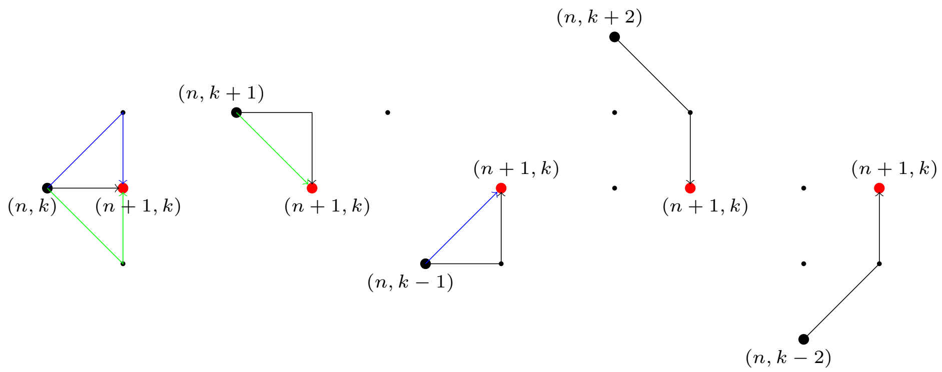


Figure 3. The five possible cases for $k \geq 2$.

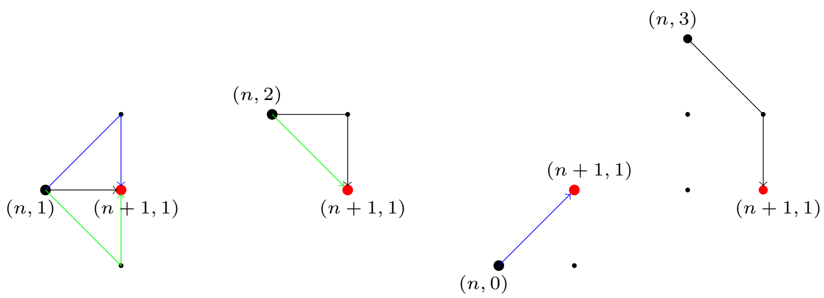


Figure 4. The four possible cases for $k = 1$.

□

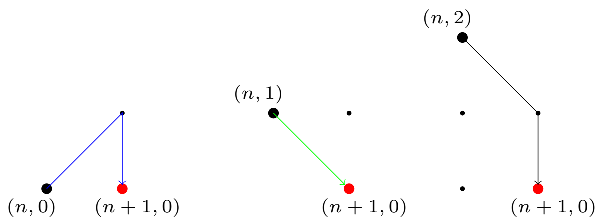


FIGURE 5. The three possible cases for $k = 0$.

From the previous theorem and Definition 3.1, we immediately obtain the following combinatorial interpretation.

Corollary 3.2. *The two-Catalan numbers* $C_n^{(2)}$ *counts the number of paths in* $\mathcal{A}_{R,2}^{Q}(n, 0)$.

4. Log-convexity of two-Catalan numbers

A sequence of nonnegative numbers $(a_n)_n$ is called log-convex if $a_i a_{i+2} \geq a_{i+1}^2$ for all $i > 0$. For more details we refer the reader to [17, 22].

By constructing injective using combinatorial interpretations by paths, Callan [10] proved the log-convexity of the Motzkin numbers, Liu and Wang [17] proved the same property for the Catalan numbers and Sun and Wang [21] also did the same for the Catalan-like numbers. Then, in a similar way Chen et al. [11] gave a combinatorial proof of the log-convexity of sequences in Riordan arrays. Motivated by these works, we give in this section an injective proof for the log-convexity of the two-Catalan numbers.

Theorem 4.1. *The sequence of two-Catalan numbers* $\left(C_n^{(2)}\right)_{n\geq 0}$ *is log-convex.*

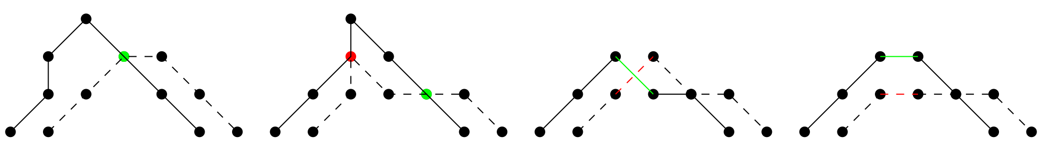
Proof. We will construct an injection ϕ from $\mathcal{A}_{R,2}^{Q}(n, 0) \times \mathcal{A}_{R,2}^{Q}(n, 0)$ to $\mathcal{A}_{R,2}^{Q}(n + 1, 0) \times \mathcal{A}_{R,2}^{Q}(n - 1, 0)$. For two paths $(P_1, P_2) \in \mathcal{A}_{R,2}^{Q}(n, 0) \times \mathcal{A}_{R,2}^{Q}(n, 0)$ such that P_1 starts at $(0, 0)$ and P_2 starts at $(1, 0)$ and in a way inspired by Callan's method, we define our "encounter" in two cases

- Not between $y = 0$ and $y = 1$, the encounter is:
  - Either a lattice point that is common between P_1 and P_2 such there is at most one vertical step linked to this point. For instance, in Figure 6a the encounter is an intersection at a lattice point, and in Figure 6b the encounter is not the red point because there are two vertical steps linked two this red point but is the green point.
  - Or the intersection of two diagonal steps as shown in Figure 6c.
  - Or a pair of flatsteps forming the top and bottom of a unit square as in Figure 6d.

- Between $y = 0$ and $y = 1$: The encounter is either as in the situation of Figure 6a or Figure 6b or Figure 6c but the situation of the Figure 6d can not be existed, this case is replaced by: the encounter is the shape of "a flatstep of the first path with a North-East step followed by a South step" as shown in Figure 8d'.

Obviously, at least one such encounter exists. Now we consider the first encounter in the two previous cases and we define the application ϕ in each case as follows

- The first encounter is not between $y = 0$ and $y = 1$:
  - In the situation of Figure 6a and Figure 6b, switch the paths to the right of the common lattice point as shown in Figure 7a and Figure 7b respectively.
  - In the situation of Figure 6c, swing each diagonal step so that it becomes an horizontal step and then the paths to the right will be switched as shown in Figure 7c.
  - In the situation of Figure 6d, change the lower horizontal step to a North-East step and the upper one to a South-East step and then the paths to the right will be switched as shown in Figure 7d.
- The first encounter is between $y = 0$ and $y = 1$:
  - If the first encounter is lattice point that is common between P_1 and P_2 such there is at most one vertical step linked to this point. This situation is the same as that of Figure 6a and Figure 6b, (switch the paths to the right of the common lattice point as shown in Figure 7a and Figure 7b respectively).
  - In the situation of Figure 8c', swing the diagonal step of the first path so that it becomes an horizontal step, and complete the diagonal step of the second path by a South step and then the paths to the right will be switched as shown in Figure 9c'.
  - In the situation of Figure 8d', change the flatstep of the first path to a South-East step, and then remove the South step of the second path then the paths to the right will be switched as shown in Figure 9d'.



a b c d

Figure 6. The first encounter is not between $y = 0$ and $y = 1$.

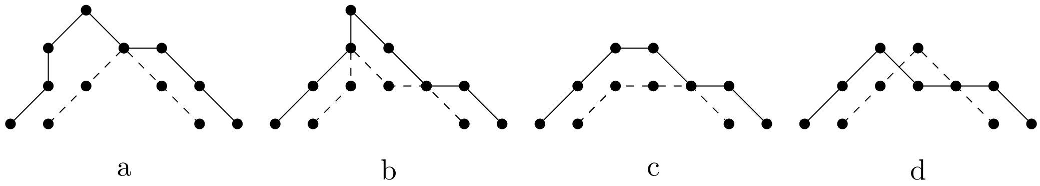


FIGURE 7. The application of ϕ_2 on the paths in Figure 6.

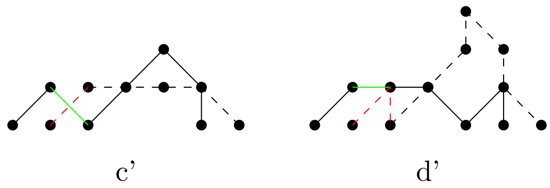


FIGURE 8. The two special cases when the first encounter is between $y = 0$ and $y = 1$.

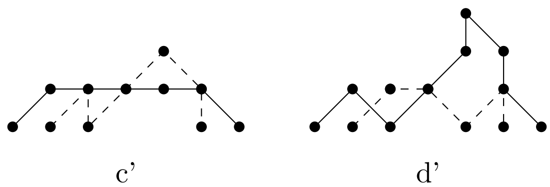


FIGURE 9. The application of ϕ on the two above special cases given in Figure 8.

In all cases, the resulting pair of paths are in $\mathcal{A}^Q_{R,2}(n+1,0) \times \mathcal{A}^Q_{R,2}(n-1,0)$. Furthermore, the location of the first encounter will remain invariant after applying ϕ, thus the mapping is reversible and then ϕ is an injection. □

By the same approach given in the previous theorem, we can obtain also the following result. Here, we omit the details for brevity.

Theorem 4.2. *The columns sequence of the Two-Catalan triangle* $\left(C^{(2)}_{n,k}\right)_{n \geq \lceil k/2 \rceil}$ *are log-convex.*

We know that the Riordan numbers of even indices equal to the two-catalan numbers. So from Theorem 4.1, we obtain the following result.

Corollary 4.1. *The sequence of Riordan numbers of even indices* $(r_{2n})_{n\geq 0}$ *is log-convex.*

5. Log-concavity of rows of two-Catalan triangle

Recall that a sequence of nonnegative numbers $(a_n)_{n\geq 0}$ is called log-concave if $a_{i-1}a_{i+1} \leq a_i^2$ for all $i > 0$, which is equivalent to having that $a_{i-1}a_{j+1} \leq a_i a_j$ for all $1 \leq i \leq j$. The log-concavity problems are appearing in combinatorics and in many other branches of mathematics and have been the the subject of many studies; see Stanley's survey article [20] and Brenti's supplement [9] for log-concavity.

In this section, we prove the log-concavity of the rows of two-Catalan triangle.

Theorem 5.1. *The rows sequence of two-Catalan triangle* $\left(C_{n,k}^{(2)}\right)_{0\leq k\leq 2n}$ *are log-concave.*

Proof. To show that $\left(C_{n,k}^{(2)}\right)_{0\leq k\leq 2n}$ is log-concave in k, it suffices to prove that

$$\left(C_{n,k}^{(2)}\right)^2 - C_{n,k-1}^{(2)}C_{n,k+1}^{(2)} \geq 0$$

for any $k \geq 0$, which will be done by induction on n. It is clear for $n = 0$. Thus, we suppose that it follows for $1 \leq n \leq m$. Then for $n = m+1$ and $0 \leq k \leq 2(m+1)$ we have that

$$\begin{aligned}
&\left(C_{m+1,k}^{(2)}\right)^2 - C_{m+1,k-1}^{(2)}C_{m+1,k+1}^{(2)} = \left[\left(C_{m,k-2}^{(2)}\right)^2 + 4\left(C_{m,k-1}^{(2)}\right)^2 + 9\left(C_{m,k}^{(2)}\right)^2 + 4\left(C_{m,k+1}^{(2)}\right)^2\right.\\
&+\left(C_{m,k+2}^{(2)}\right)^2 + 4C_{m,k-2}^{(2)}C_{m,k-1}^{(2)} + 6C_{m,k-2}^{(2)}C_{m,k}^{(2)} + 4C_{m,k-2}^{(2)}C_{m,k+1}^{(2)} + 2C_{m,k-2}^{(2)}C_{m,k+2}^{(2)}\\
&+12C_{m,k-1}^{(2)}C_{m,k}^{(2)} + 8C_{m,k-1}^{(2)}C_{m,k+1}^{(2)} + 4C_{m,k-1}^{(2)}C_{m,k+2}^{(2)} + 12C_{m,k}^{(2)}C_{m,k+1}^{(2)}\\
&\left.+6C_{m,k}^{(2)}C_{m,k+2}^{(2)} + 4C_{m,k+1}^{(2)}C_{m,k+2}^{(2)}\right] - \left[C_{m,k-3}^{(2)}C_{m,k-1}^{(2)} + 4C_{m,k-2}^{(2)}C_{m,k}^{(2)} + 9C_{m,k-1}^{(2)}C_{m,k+1}^{(2)}\right.\\
&+4C_{m,k}^{(2)}C_{m,k+2}^{(2)} + C_{m,k+1}^{(2)}C_{m,k+3}^{(2)} + 2C_{m,k-3}^{(2)}C_{m,k}^{(2)} + 2C_{m,k-2}^{(2)}C_{m,k-1}^{(2)} + 3C_{m,k-3}^{(2)}C_{m,k+1}^{(2)}\\
&+3\left(C_{m,k-1}^{(2)}\right)^2 + 2C_{m,k-3}^{(2)}C_{m,k+2}^{(2)} + 2C_{m,k-1}^{(2)}C_{m,k}^{(2)} + C_{m,k-3}^{(2)}C_{m,k+3}^{(2)} + C_{m,k-1}^{(2)}C_{m,k+1}^{(2)}\\
&+6C_{m,k-2}^{(2)}C_{m,k+1}^{(2)} + 6C_{m,k-1}^{(2)}C_{m,k}^{(2)} + 4C_{m,k-2}^{(2)}C_{m,k+2}^{(2)} + 4\left(C_{m,k}^{(2)}\right)^2 + 2C_{m,k-2}^{(2)}C_{m,k+3}^{(2)}\\
&+2C_{m,k}^{(2)}C_{m,k+1}^{(2)} + 6C_{m,k-1}^{(2)}C_{m,k+2}^{(2)} + 6C_{m,k}^{(2)}C_{m,k+1}^{(2)} + 3C_{m,k-1}^{(2)}C_{m,k+3}^{(2)} + 3\left(C_{m,k+1}^{(2)}\right)^2\\
&\left.+2C_{m,k}^{(2)}C_{m,k+3}^{(2)} + 2C_{m,k+1}^{(2)}C_{m,k+2}^{(2)}\right].
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
&\left(C^{(2)}_{m+1,k}\right)^2 - C^{(2)}_{m+1,k-1}C^{(2)}_{m+1,k+1} = \left(\left(C^{(2)}_{m,k-2}\right)^2 - C^{(2)}_{m,k-3}C^{(2)}_{m,k-1}\right)\\
&+\left(\left(C^{(2)}_{m,k-1}\right)^2 - C^{(2)}_{m,k-2}C^{(2)}_{m,k}\right) + \left(\left(C^{(2)}_{m,k+1}\right)^2 - C^{(2)}_{m,k}C^{(2)}_{m,k+2}\right)\\
&+5\left(\left(C^{(2)}_{m,k}\right)^2 - C^{(2)}_{m,k-1}C^{(2)}_{m,k+1}\right) + \left(\left(C^{(2)}_{m,k+2}\right)^2 - C^{(2)}_{m,k+1}C^{(2)}_{m,k+3}\right)\\
&+2\left(C^{(2)}_{m,k-2}C^{(2)}_{m,k-1} - C^{(2)}_{m,k-3}C^{(2)}_{m,k}\right) + 3\left(C^{(2)}_{m,k-2}C^{(2)}_{m,k} - C^{(2)}_{m,k-3}C^{(2)}_{m,k+3}\right)\\
&+2\left(C^{(2)}_{m,k-2}C^{(2)}_{m,k+1} - C^{(2)}_{m,k-3}C^{(2)}_{m,k+2}\right) + 4\left(C^{(2)}_{m,k-1}C^{(2)}_{m,k} - C^{(2)}_{m,k-2}C^{(2)}_{m,k+1}\right)\\
&+\left(C^{(2)}_{m,k-2}C^{(2)}_{m,k+2} - C^{(2)}_{m,k-3}C^{(2)}_{m,k+3}\right) + 3\left(C^{(2)}_{m,k-1}C^{(2)}_{m,k+1} - C^{(2)}_{m,k-2}C^{(2)}_{m,k+2}\right)\\
&+2\left(C^{(2)}_{m,k-1}C^{(2)}_{m,k+2} - C^{(2)}_{m,k-2}C^{(2)}_{m,k+3}\right) + 4\left(C^{(2)}_{m,k}C^{(2)}_{m,k+1} - C^{(2)}_{m,k-1}C^{(2)}_{m,k+2}\right)\\
&+3\left(C^{(2)}_{m,k}C^{(2)}_{m,k+2} - C^{(2)}_{m,k-1}C^{(2)}_{m,k+3}\right) + 2\left(C^{(2)}_{m,k+1}C^{(2)}_{m,k+2} - C^{(2)}_{m,k}C^{(2)}_{m,k+3}\right) \geq 0
\end{aligned}$$

since $\left(C^{(2)}_{m,k}\right)_{0\leq k\leq 2m}$ is log-concave. This completes the proof. □

6. Concluding remarks and open problems

In this paper, we have studied the log-convexity of two-Catalan numbers by using the combinatorial interpretation proposed in Section 3. It is natural wonder if it is possible to prove the log-convexity the s-Catalan for any positive integer s, by perhaps looking for an appropriate combinatorial interpretation that makes the proof easier.

Open Problem Are the s-Catalan numbers log-convex for any positive integer $s \geq 3$?

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