MANN-TYPE INERTIAL SUBGRADIENT EXTRAGRADIENT METHODS FOR BILEVEL EQUILIBRIUM PROBLEMS

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In this paper, we introduce and analyze two Mann-type implicit inertial subgradient extragradient algorithms for solving the monotone bilevel equilibrium problem with a general system of variational inclusions and a common fixed-point problem of a finite family of strict pseudocontraction mappings and an asymptotically nonexpansive mapping constraints. Some strong convergence theorems for the proposed algorithms are established under the suitable assumptions.

Keywords: inertial subgradient extragradient method, bilevel equilibrium problem, variational inclusions, fixed point.

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1. Introduction

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with induced norm $\|\cdot\|$. Let the nonempty subset $C \subset \mathcal{H}$ be closed and convex. A mapping $T : C \to C$ is known as ζ-strictly pseudocontractive if there exists a constant $\zeta \in [0, 1)$ such that $\|Tx - Ty\|^2 \le \|x - y\|^2 + \zeta\|(I - T)x - (I - T)y\|^2, \forall x, y \in C$. A mapping $T : C \to C$ is known as asymptotically nonexpansive, if there exists a sequence $\{\theta_k\} \subset [0, \infty)$ such that $\|T^k x - T^k y\| \le (1 + \theta_k)\|x - y\|, \forall x, y \in C, k \ge 1$, with $\lim_{k\to\infty} \theta_k = 0$. We denote by $\mathrm{Fix}(T)$ the fixed-point set of the mapping T. Let A be a self-mapping on $\mathcal{H}$. The classical variational inequality problem ([36, 37]) (VIP) is to find $x^* \in C$ s.t. $\langle Ax^*, y - x^*\rangle \ge 0, \forall y \in C$. The solution set of the VIP is denote by $\mathrm{VI}(C, A)$. Let the Ω denote the common solution set of the fixed-point problem (FPP) of asymptotically nonexpansive mapping $T : C \to C$ with $\{\theta_k\}$ and the variational inequality problems (VIPs) for two inverse-strongly monotone mappings F_1, F_2. Suppose that the bifunction $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ satisfies $\Phi(x, x) = 0, \forall x \in C$. Consider the equilibrium problem $(\mathrm{EP}(C, \Phi))$ which is to find $x^* \in C$ such that

$$\Phi(x^*, y) \ge 0, \ \forall y \in C. \tag{1}$$

The solution set of $\mathrm{EP}(C, \Phi)$ is denoted by $\mathrm{Sol}(C, \Phi)$. It is well known that the $\mathrm{EP}(C, \Phi)$ as a unified model plays an important role in the research of several problems, e.g., variational inequality problems ([5, 8, 27, 28, 30, 32, 40, 44]), optimization problems ([14, 18, 19, 24, 25, 38, 41, 43]), split problems ([12, 13, 31, 35]), saddle point problems, complementarity problems, fixed point problems ([20–23, 26]), Nash equilibrium problems, etc. The $\mathrm{EP}(C, \Phi)$ and its extended versions have been widely studied by many authors; see [10, 17, 29, 33, 34, 39, 42] and references therein. Anh and An [2] considered the monotone bilevel equilibrium

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problem (MBEP) with the fixed-point problem constraint:

$$\text{Find } x^* \in \Omega \text{ such that } \Psi(x^*, y) \geq 0,\ \forall y \in \Omega, \tag{2}$$

where $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ such that $\Psi(x, x) = 0, \forall x \in C$ and $\Omega = \mathrm{Sol}(C, \Phi) \cap \mathrm{Fix}(T)$.

Choose the parameter sequences $\{\lambda_n\}$ and $\{\beta_n\}$ such that

$$\begin{cases} \{\lambda_n\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}),\ \lim_{n\to\infty} \lambda_n = \lambda, \beta_n \downarrow 0,\ 2\beta_n\eta - \beta_n^2 \Upsilon^2 < 1, \\ \sum_{n=0}^{\infty} \beta_n = +\infty, 0 < \tau < \min\{\eta, \Upsilon\},\ 0 < \beta_n < \min\{\frac{1}{\tau}, \frac{2\eta-2\tau}{\Upsilon^2-\tau^2}, \frac{2\eta}{\Upsilon^2}\}, \end{cases} \tag{3}$$

where Υ is a constant associated with Ψ.

Let $F_1, F_2 : \mathcal{H} \to \mathcal{H}$ be single-valued mappings and $B_1, B_2 : C \to 2^{\mathcal{H}}$ be multi-valued mappings with $B_j y \neq \emptyset, \forall y \in C, j = 1, 2$. Consider the general system of variational inclusions (GSVI), which is to find $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} 0 \in \lambda_1(F_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \lambda_2(F_2 x^* + B_2 y^*) + y^* - x^*. \end{cases} \tag{4}$$

In particular, if $F_1 = F_2 = A,\ B_1 = B_2 = B$ and $x^* = y^*$, then problem (4) reduces to the variational inclusion (VI) ([6]). It is known that problem (4) has been transformed into a fixed point problem in the following way.

Proposition 1.1 ([7]). *Suppose that the mappings $B_1, B_2 : C \to 2^{\mathcal{H}}$ both are maximal monotone. Then for given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (4) if and only if $x^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed-point set of the mapping $G := J_{\lambda_1}^{B_1}(I-\lambda_1 F_1) J_{\lambda_2}^{B_2}(I-\lambda_2 F_2)$, and $y^* = J_{\lambda_2}^{B_2}(I - \lambda_2 F_2)x^*$.*

In this paper, we introduce and analyze two iterative algorithms for solving the monotone bilevel equilibrium problem (MBEP) with a general system of variational inclusions (GSVI) and a common fixed-point problem of a finite family of strict pseudocontraction mappings and an asymptotically nonexpansive mapping (CFPP) constraints, i.e., a strongly monotone equilibrium problem $\mathrm{EP}(\Omega, \Psi)$ over the common solution set Ω of another monotone equilibrium problem $\mathrm{EP}(C, \Phi)$, the GSVI and the CFPP. Some strong convergence results for the proposed algorithms are established under the suitable assumptions. Our results improve and extend some corresponding results in the earlier and very recent literature; see e.g., [2, 5, 30].

2. **Preliminaries**

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. In the following, we denote by " $\rightharpoonup$ " weak convergence and by " $\to$ " strong convergence. A bifunction $\Psi : C \times C \to \mathcal{R}$ is said to be

(i) η-strongly monotone, if $\Psi(x, y) + \Psi(y, x) \leq -\eta\|x - y\|^2, \forall x, y \in C$;
(ii) monotone, if $\Psi(x, y) + \Psi(y, x) \leq 0, \forall x, y \in C$;
(iii) Lipschitz-type continuous with constants $c_1, c_2 > 0$, if $\Psi(x, y) + \Psi(y, z) \geq \Psi(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \forall x, y, z \in C$.

Recall that a mapping $F : C \to \mathcal{H}$ is said to be

(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ s.t. $\|Fx-Fy\| \leq L\|x-y\|, \forall x, y \in C$;
(ii) monotone if $\langle Fx - Fy, x - y\rangle \geq 0, \forall x, y \in C$;
(iii) pseudomonotone if $\langle Fx, y - x\rangle \geq 0 \Rightarrow \langle Fy, y - x\rangle \geq 0, \forall x, y \in C$;
(iv) η-strongly monotone if $\exists \eta > 0$ s.t. $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \forall x, y \in C$;
(v) α-inverse-strongly monotone if $\exists \alpha > 0$ s.t. $\langle Fx - Fy, x - y\rangle \geq \alpha\|Fx - Fy\|^2, \forall x, y \in C$.

Recall that the mapping $T : C \to C$ is a ζ-strict pseudocontraction for some $\zeta \in [0, 1)$ if and only if the inequality holds $\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\zeta}{2}\|(I - T)x - (I - T)y\|^2, \forall x, y \in C$. If T is a ζ-strictly pseudocontractive mapping, then T satisfies Lipschitz condition

$\|Tx - Ty\| \leq \frac{1+\zeta}{1-\zeta}\|x - y\|, \forall x, y \in C$. For each point $x \in \mathcal{H}$, we know that there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, $\forall y \in C$. The mapping P_C is said to be the metric projection of $\mathcal{H}$ onto C. Recall that the following statements hold:

(i) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in \mathcal{H}$;
(ii) $\langle x - P_C x, y - P_C x\rangle \leq 0$, $\forall x \in \mathcal{H}, y \in C$;
(iii) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$, $\forall x \in \mathcal{H}, y \in C$;
(iv) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, $\forall x, y \in \mathcal{H}$;
(v) $\|sx + (1 - s)y\|^2 = s\|x\|^2 + (1 - s)\|y\|^2 - s(1 - s)\|x - y\|^2$, $\forall x, y \in \mathcal{H}, s \in [0, 1]$.

Lemma 2.1 ([1]). *Let $T : C \to C$ be a ζ-strict pseudocontraction. Then $I - T$ is demiclosed.*

Lemma 2.2 ([30]). *Let $T : C \to C$ be a ζ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma + \delta)\zeta \leq \gamma$. Then $\|\gamma(x - y) + \delta(Tx - Ty)\| \leq (\gamma + \delta)\|x - y\|, \forall x, y \in C$.*

Let $B : C \to 2^{\mathcal{H}}$ be a set-valued operator with $Bx \neq \emptyset, \forall x \in C$. B is said to be monotone ([11]) if for each $x, y \in C$, one has $\langle u - v, x - y\rangle \geq 0, \forall u \in Bx, v \in By$. Also, B is said to be maximal monotone if $(I + \lambda B)C = \mathcal{H}$ for all $\lambda > 0$. For a monotone operator B, we define the mapping $J^B_\lambda : (I + \lambda B)C \to C$ by $J^B_\lambda = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J^B_λ is called the resolvent of B for $\lambda > 0$. Let $F : \mathcal{H} \to \mathcal{H}$ be an α-inverse-strongly monotone mapping and $B : C \to 2^{\mathcal{H}}$ be a maximal monotone operator. In the sequel, we shall use the notation $T_\lambda := J^B_\lambda(I - \lambda F) = (I + \lambda B)^{-1}(I - \lambda F), \forall \lambda > 0$.

Proposition 2.1 ([15]). *Let $B : C \to 2^{\mathcal{H}}$ be a maximal monotone operator. Then the following statements hold: (i) the resolvent identity: $J^B_\lambda x = J^B_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J^B_\lambda x)$, $\forall \lambda, \mu > 0$, $x \in \mathcal{H}$; (ii) if J^B_λ is a resolvent of B for $\lambda > 0$, then J^B_λ is a firmly nonexpansive mapping with $\mathrm{Fix}(J^B_\lambda) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$.*

Proposition 2.2 ([15]). *The following statements hold: (i) $\mathrm{Fix}(T_\lambda) = (F + B)^{-1}0, \forall \lambda > 0$; (ii) $\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ for $0 < \lambda \leq r$ and $y \in C$.*

Lemma 2.3. *Let the mapping $F : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone. Then, for a given $\lambda \geq 0$, $\|(I - \lambda F)x - (I - \lambda F)y\|^2 \leq \|x - y\|^2 - \lambda(2\alpha - \lambda)\|Fx - Fy\|^2, \forall x, y \in \mathcal{H}$. In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda F$ is nonexpansive.*

Lemma 2.4. *Let the operators $B_1, B_2 : C \to 2^{\mathcal{H}}$ be both maximal monotone. Let the mappings $F_1, F_2 : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : \mathcal{H} \to C$ be defined as $G := J^{B_1}_{\lambda_1}(I - \lambda_1 F_1)J^{B_2}_{\lambda_2}(I - \lambda_2 F_2)$. If $0 < \lambda_1 \leq 2\alpha$ and $0 < \lambda_2 \leq 2\beta$, then $G : \mathcal{H} \to C$ is nonexpansive.*

Lemma 2.5. *The inequality holds: $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ $\forall x, y \in \mathcal{H}$.*

Lemma 2.6 ([9]). *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if $\{u^k\}$ is a sequence in C such that $u^k \rightharpoonup u \in C$ and $(I - T)u^k \to 0$, then $(I - T)u = 0$, where I is the identity mapping of X.*

Lemma 2.7 ([16]). *Let $\{\Gamma_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{k_j}\}$ of $\{\Gamma_k\}$ which satisfies $\Gamma_{k_j} < \Gamma_{k_j+1}$ for each integer $j \geq 1$. Define the sequence $\{\tau(k)\}_{k \geq k_0}$ of integers as follows: $\tau(k) = \max\{j \leq k : \Gamma_j < \Gamma_{j+1}\}$, where integer $k_0 \geq 1$ such that $\{j \leq k_0 : \Gamma_j < \Gamma_{j+1}\} \neq \emptyset$. Then, the following hold: (i) $\tau(k_0) \leq \tau(k_0 + 1) \leq \cdots$ and $\tau(k) \to \infty$; (ii) $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$ and $\Gamma_k \leq \Gamma_{\tau(k)+1}, \forall k \geq k_0$.*

The normal cone $N_C(x)$ of C at $x \in C$ is defined as $N_C(x) = \{z \in \mathcal{H} : \langle z, y - x\rangle \leq 0, \forall y \in C\}$. The subdifferential of a convex function $g : C \to \mathcal{R} \cup \{+\infty\}$ at $x \in C$ is defined by $\partial g(x) = \{z \in \mathcal{H} : g(y) - g(x) \geq \langle z, y - x\rangle,\ \forall y \in C\}$.

In this paper, we are committed to finding a solution $x^* \in \mathrm{Sol}(\Omega, \Psi)$ of the problem $\mathrm{EP}(\Omega, \Psi)$, where $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi)$ with $T_0 := T$. We assume always that $T : \mathcal{H} \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_k\}$ and $T_i : C \to C$ is a ζ_i-strict pseudocontraction for $i = 1, ..., N$ such that $\zeta := \max\{\zeta_i : 1 \leq i \leq N\}$. $B_1, B_2 : C \to 2^{\mathcal{H}}$ are two maximal monotone operators, and $F_1, F_2 : \mathcal{H} \to \mathcal{H}$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. $G : \mathcal{H} \to C$ is defined as $Gx = J_{\lambda_1}^{B_1}(I - \lambda_1 F_1) J_{\lambda_2}^{B_2}(I - \lambda_2 F_2)x\ \forall x \in \mathcal{H}$ where $0 < \lambda_1 < 2\alpha$ and $0 < \lambda_2 < 2\beta$. Choose the sequences $\{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0, 1)$, and positive sequences $\{\alpha_k\}$, $\{\varepsilon_k\}, \{s_k\}$ such that

(H1) $\beta_k + \gamma_k + \delta_k = 1\ \forall k \geq 1$ and $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$;
(H2) $(\gamma_k + \delta_k)\zeta \leq \gamma_k\ \forall k \geq 1$, $0 < \liminf_{k\to\infty} \delta_k$, $\sum_{k=1}^{\infty} \varepsilon_k < \infty$ and $\sum_{k=1}^{\infty} \theta_k < \infty$;
(H3) $\lim_{k\to\infty} s_k = 0$, $\sup_{k\geq 1} \frac{\varepsilon_k}{s_k} < \infty$, $\lim_{k\to\infty} \frac{\theta_k}{s_k} = 0$ and $\sum_{k=1}^{\infty} s_k = \infty$;
(H4) $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ and $\lim_{k\to\infty} \alpha_k = \tilde{\alpha}$;
(H5) $2s_k\nu - s_k^2 \Upsilon^2 < 1$, $0 < \lambda < \min\{\nu, \Upsilon\}$ and $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu-2\lambda}{\Upsilon^2-\lambda^2}, \frac{2\nu}{\Upsilon^2}\}$.

Write $T_k := T_{k\mathrm{mod}N}$ for integer $k \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_k = T_N$ if $q = 0$ and $T_k = T_q$ if $0 < q < N$.

Algorithm 2.1. *Let $x^0, x^1 \in C$ be arbitrary. The sequences $\{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0, 1)$, and positive sequences $\{\alpha_k\}, \{\varepsilon_k\}, \{s_k\}$ satisfy conditions (H1)-(H5). Calculate x^{k+1} as follows:*
Step 1. Compute $\tau^k = x^k + \varepsilon_k(x^k - x^{k-1})$ and $y^k = \mathrm{argmin}\{\alpha_k \Phi(\tau^k, y) + \frac{1}{2}\|y - \tau^k\|^2 : y \in C\}$.
Step 2. Choose $w^k \in \partial_2 \Phi(\tau^k, y^k)$. Compute $C_k = \{v \in \mathcal{H} : \langle \tau^k - \alpha_k w^k - y^k, v - y^k\rangle \leq 0\}$ and $z^k = \mathrm{argmin}\{\alpha_k \Phi(y^k, z) + \frac{1}{2}\|z - \tau^k\|^2 : z \in C_k\}$.
Step 3. Compute $\mu^k = \beta_k T^k z^k + \gamma_k p^k + \delta_k T_k p^k$, $v^k = J_{\lambda_2}^{B_2}(\mu^k - \lambda_2 F_2 \mu^k)$ and $p^k = J_{\lambda_1}^{B_1}(v^k - \lambda_1 F_1 v^k)$.
Step 4. Compute $x^{k+1} = \mathrm{argmin}\{s_k \Psi(\mu^k, t) + \frac{1}{2}\|t - \mu^k\|^2 : t \in C\}$. Set $k := k + 1$ and return to Step 1.

Proposition 2.3 ([4])**.** *Let C be a convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathcal{R} \cup \{+\infty\}$ be subdifferentiable. Then, $\hat{x}$ is a solution to the following convex minimization problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(\hat{x}) + N_C(\hat{x})$, where ∂g denotes the subdifferential of g.*

Proposition 2.4 ([3])**.** *Let X and Y be two sets, $\mathcal{G}$ be a set-valued map from Y to X, and W be a real valued function defined on $X \times Y$. The marginal function M is defined as*

$$M(y) = \{x^* \in \mathcal{G}(y) : W(x^*, y) = \sup\{W(x, y) : x \in \mathcal{G}(y)\}\}.$$

If W and $\mathcal{G}$ are continuous, then M is upper semicontinuous.

Next, we assume that two bifunctions $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ and $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ satisfy the following conditions:

Ass_Φ:
(Φ_1) $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi) \neq \emptyset$ with $T_0 := T$.
(Φ_2) Φ is monotone and Lipschitz-type continuous with constants $c_1, c_2 > 0$, and Φ is weakly continuous, i.e., $\{x^k \rightharpoonup \bar{x}$ and $y^k \rightharpoonup \bar{y}\} \Rightarrow \{\Phi(x^k, y^k) \to \Phi(\bar{x}, \bar{y})\}$.

Ass_Ψ:
(Ψ_1) Ψ is ν-strongly monotone and weakly continuous.
(Ψ_2) There exist the mappings $\bar{\Psi}_i, \hat{\psi}_i : C \times C \to \mathcal{H}$ for each $i \in \{1, ..., m\}$, such that for all $u, v, x, y, z \in C$ the following hold:
(a) $\bar{\Psi}_i(x, y) + \bar{\Psi}_i(y, x) = 0$, $\|\bar{\Psi}_i(x, y)\| \leq \bar{L}_i\|x - y\|$.

(b) $\hat{\psi}_i(x,x) = 0$ and $\|\hat{\psi}_i(u,v) - \hat{\psi}_i(x,y)\| \le \hat{L}_i\|(u-v)-(x-y)\|$.
(c) $\Psi(x,y) + \Psi(y,z) \ge \Psi(x,z) + \sum_{i=1}^{m}\langle \bar{\Psi}_i(x,y), \hat{\psi}_i(y,z)\rangle$.

(Ψ_3) For any sequence $\{y^k\} \subset C$ such that $y^k \to d$, we have $\limsup_{k\to\infty} \frac{|\Psi(d,y^k)|}{\|y^k-d\|} < +\infty$.

3. Main Results

In this section, we consider and analyze two Mann-type implicit inertial subgradient extragradient algorithms for solving the MBEP with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem EP(Ω,Ψ) over the common solution set Ω of another monotone equilibrium problem EP(C,Φ), the GSVI (4) and the CFPP, where $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C,\Phi)$ with $T_0 := T$.

Theorem 3.1. *Suppose that $\{x^k\}$ is the sequence defined by Algorithm 2.1, such that $\|T^k x^k - T^{k+1}x^k\| \to 0$ as $k \to \infty$. Let the bifunctions Ψ, Φ satisfy the assumptions* $\mathbf{Ass}_{\Phi}$-$\mathbf{Ass}_{\Psi}$*, and assume that the conditions (H1)-(H5) hold. Then $x^k \to x^* \in \Omega \Leftrightarrow x^k - x^{k+1} \to 0$, where $x^* \in \Omega$ is a unique solution to the problem EP(Ω,Ψ).*

Proof. For each $k \ge 1$, let $\Gamma_k : \mathcal{H} \to C$ be a mapping defined by $\Gamma_k x := \beta_k T^k z^k + \gamma_k Gx + \delta_k T_k Gx\ \forall x \in \mathcal{H}$. Note that the mapping $G : \mathcal{H} \to C$ is defined as $G = J_{\lambda_1}^{B_1}(I - \lambda_1 F_1)J_{\lambda_2}^{B_2}(I - \lambda_2 F_2)$, where $\lambda_1 \in (0,2\alpha)$ and $\lambda_2 \in (0,2\beta)$. Then, by Lemma 2.4, we know that G is nonexpansive. Since $(\gamma_k+\delta_k)\zeta \le \gamma_k$, by Lemma 2.2 we obtain that for all $x,y \in \mathcal{H}$, $\|\Gamma_k x - \Gamma_k y\| \le (\gamma_k+\delta_k)\|Gx - Gy\| \le (1-\beta_k)\|x-y\|$. Hence, by the Banach contraction mapping principle, we deduce from $\{\beta_k\} \subset (0,1)$ that for each $k \ge 1$, there exists a unique element $\mu^k \in C$ such that

$$\mu^k = \beta_k T^k z^k + \gamma_k G\mu^k + \delta_k T_k G\mu^k. \tag{5}$$

Choose an element $p \in \Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C,\Phi)$ arbitrarily. Since $\lim_{k\to\infty} \frac{\theta_k}{s_k} = 0$, we may assume, without loss of generality, that $\theta_k \le \frac{1}{2}\lambda s_k$ for all $k \ge 1$. We divide the rest of the proof into several steps as follows:

Step 1. We show that the following inequality holds

$$\|z^k - p\|^2 \le \|\tau^k - p\|^2 - (1-2\alpha_k c_1)\|y^k - \tau^k\|^2 - (1-2\alpha_k c_2)\|z^k - y^k\|^2,\ \forall k \ge 1.$$

Indeed, by Proposition 2.3, we know that for $y^k = \mathrm{argmin}\{\alpha_k \Phi(\tau^k, y) + \frac{1}{2}\|y - \tau^k\|^2 : y \in C\}$, there exists $w^k \in \partial_2\Phi(\tau^k, y^k)$ such that $\alpha_k w^k + y^k - \tau^k \in -N_C(y^k)$, which hence yields $\langle \alpha_k w^k + y^k - \tau^k, x - y^k\rangle \ge 0, \forall x \in C$. From the definition of $w^k \in \partial_2\Phi(\tau^k, y^k)$, it follows that

$$\alpha_k[\Phi(\tau^k, x) - \Phi(\tau^k, y^k)] \ge \langle \alpha_k w^k, x - y^k\rangle,\ \forall x \in \mathcal{H}. \tag{6}$$

Adding the last two inequalities, we get

$$\alpha_k[\Phi(\tau^k, x) - \Phi(\tau^k, y^k)] + \langle y^k - \tau^k, x - y^k\rangle \ge 0,\ \forall x \in C. \tag{7}$$

It follows from $z^k \in C_k$ and the definition of C_k that $\langle \tau^k - \alpha_k w^k - y^k, v - y^k\rangle \le 0$, and hence

$$\alpha_k\langle w^k, z^k - y^k\rangle \ge \langle \tau^k - y^k, z^k - y^k\rangle. \tag{8}$$

Putting $x = z^k$ in (6), we get $\alpha_k[\Phi(\tau^k, z^k) - \Phi(\tau^k, y^k)] \ge \alpha_k\langle w^k, z^k - y^k\rangle$. Adding (8) and the last inequality, we have

$$\alpha_k[\Phi(\tau^k, z^k) - \Phi(\tau^k, y^k)] \ge \langle \tau^k - y^k, z^k - y^k\rangle. \tag{9}$$

By Proposition 2.3, we know that for $z^k = \mathrm{argmin}\{\alpha_k\Phi(y^k, y) + \frac{1}{2}\|y - \tau^k\|^2 : y \in C_k\}$, there exist $h^k \in \partial_2\Phi(y^k, z^k)$ and $r^k \in N_{C_k}(z^k)$ such that $\alpha_k h^k + z^k - \tau^k + r^k = 0$. So, we infer that $\alpha_k\langle h^k, y - z^k\rangle \ge \langle \tau^k - z^k, y - z^k\rangle, \forall y \in C_k$, and $\Phi(y^k, y) - \Phi(y^k, z^k) \ge \langle h^k, y - z^k\rangle, \forall y \in \mathcal{H}$. Putting $y = p \in C \subset C_k$ in two last inequalities and later adding them, we get $\alpha_k[\Phi(y^k,p)-\Phi(y^k,z^k)] \geq \langle \tau^k - z^k, p - z^k\rangle$. Therefore, $-\alpha_k\Phi(y^k,z^k) \geq \langle \tau^k - z^k, p - z^k\rangle$. Combining this and the following Lipschitz-type continuity of Φ, we obtain that

$$\langle \tau^k - z^k, z^k - p\rangle \geq \alpha_k[\Phi(\tau^k,z^k) - \Phi(\tau^k,y^k)] - \alpha_k c_1\|\tau^k - y^k\|^2 - \alpha_k c_2\|y^k - z^k\|^2.$$

This together with (9), implies that

$$\langle \tau^k - z^k, z^k - p\rangle \geq \langle \tau^k - y^k, z^k - y^k\rangle - \alpha_k c_1\|\tau^k - y^k\|^2 - \alpha_k c_2\|y^k - z^k\|^2. \tag{10}$$

Therefore, applying the equality

$$\langle u,v\rangle = \frac{1}{2}(\|u+v\|^2 - \|u\|^2 - \|v\|^2),\ \forall u,v \in \mathcal{H}, \tag{11}$$

for $\langle \tau^k - z^k, z^k - p\rangle$ and $\langle y^k - \tau^k, z^k - y^k\rangle$ in (10), we obtain the desired result.
Step 2. We show that the following inequality holds

$$\|x^{k+1} - x\|^2 \leq \|\mu^k - x\|^2 - \|x^{k+1} - \mu^k\|^2 + 2s_k[\Psi(\mu^k,x) - \Psi(\mu^k,x^{k+1})],\ \forall x \in C.$$

Indeed, since $x^{k+1} = \operatorname{argmin}\{s_k\Psi(\mu^k,t)+\frac{1}{2}\|t-\mu^k\|^2 : t \in C\}$, there exists $m^k \in \partial_2\Psi(\mu^k,x^{k+1})$ such that $0 \in s_k m^k + x^{k+1} - \mu^k + N_C(x^{k+1})$. By the definition of normal cone N_C and the subgradient m^k, we get $\langle s_k m^k + x^{k+1} - \mu^k, x - x^{k+1}\rangle \geq 0, \forall x \in C$ and $s_k[\Psi(\mu^k,x) - \Psi(\mu^k,x^{k+1})] \geq \langle s_k m^k, x - x^{k+1}\rangle, \forall x \in C$. Adding the last two inequalities, we get

$$2s_k[\Psi(\mu^k,x) - \Psi(\mu^k,x^{k+1})] + 2\langle x^{k+1} - \mu^k, x - x^{k+1}\rangle \geq 0,\ \forall x \in C. \tag{12}$$

Putting $u = x^{k+1} - \mu^k$ and $v = x - x^{k+1}$ in (11), we get

$$2s_k[\Psi(x^{k+1},x) - \Psi(\mu^k,x^{k+1})] + \|\mu^k - x\|^2 - \|x^{k+1} - \mu^k\|^2 - \|x^{k+1} - x\|^2 \geq 0,\ \forall x \in C.$$

This attains the desired result.
Step 3. We show that if x^* is a solution of the MBEP with the GSVI and CFPP constraints, then $\|x^{k+1} - \mu_*^k\| \leq \eta_k\|\mu^k - x^*\| \leq (1-\lambda s_k)\|\mu^k - x^*\|$, where $\mu_*^k = \operatorname{argmin}\{s_k\Psi(x^*,v)+\frac{1}{2}\|v - x^*\|^2 : v \in C\}$, $\eta_k = \sqrt{1 - 2s_k\nu + s_k^2\Upsilon^2}$, $0 < \lambda < \min\{\nu, \Upsilon\}$, $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{\Upsilon^2 - \lambda^2}\}$, and $\Upsilon = \sum_{i=1}^m \bar{L}_i\hat{L}_i$. Indeed, put $\mu_*^k = \operatorname{argmin}\{s_k\Psi(x^*,v) + \frac{1}{2}\|v - x^*\|^2 : v \in C\}$. By the similar arguments to those of (12), we also get

$$s_k[\Psi(x^*,x) - \Psi(x^*,\mu_*^k)] + \langle \mu_*^k - x^*, x - \mu_*^k\rangle \geq 0 \quad \forall x \in C. \tag{13}$$

Setting $x = \mu_*^k \in C$ in (12) and $x = x^{k+1} \in C$ in (13), respectively, we obtain that $s_k[\Psi(\mu^k,\mu_*^k) - \Psi(\mu^k,x^{k+1})] + \langle x^{k+1} - \mu^k, \mu_*^k - x^{k+1}\rangle \geq 0$ and $s_k[\Psi(x^*,x^{k+1}) - \Psi(x^*,\mu_*^k)] + \langle \mu_*^k - x^*, x^{k+1} - \mu_*^k\rangle \geq 0$. Adding the last two inequalities, we have

$$\begin{aligned} 0 &\leq 2s_k[\Psi(\mu^k,\mu_*^k) - \Psi(\mu^k,x^{k+1}) + \Psi(x^*,x^{k+1}) - \Psi(x^*,\mu_*^k)] + \|\mu^k - x^*\|^2 \\ &\quad - \|x^{k+1} - \mu^k - \mu_*^k + x^*\|^2 - \|x^{k+1} - \mu_*^k\|^2, \end{aligned} \tag{14}$$

where the last equality follows directly from (11). Note that, under assumption $\mathbf{Ass}_\Psi(\Psi_2)$, it follows that $\Psi(\mu^k,\mu_*^k) - \Psi(x^*,\mu_*^k) \leq \Psi(\mu^k,x^*) - \sum_{i=1}^m \langle \bar{\Psi}_i(\mu^k,x^*), \hat{\psi}_i(x^*,\mu_*^k)\rangle$. Hence, $\Psi(x^*,x^{k+1}) - \Psi(\mu^k,x^{k+1}) \leq \Psi(x^*,\mu^k) - \sum_{i=1}^m \langle \bar{\Psi}_i(x^*,\mu^k), \hat{\psi}_i(\mu^k,x^{k+1})\rangle$. Therefore, we have

$$\begin{aligned} &\Psi(\mu^k,\mu_*^k) - \Psi(\mu^k,x^{k+1}) + \Psi(x^*,x^{k+1}) - \Psi(x^*,\mu_*^k) \\ &\leq \Psi(\mu^k,x^*) + \Psi(x^*,\mu^k) - \sum_{i=1}^m \langle \bar{\Psi}_i(\mu^k,x^*), \hat{\psi}_i(x^*,\mu_*^k)\rangle - \sum_{i=1}^m \langle \bar{\Psi}_i(x^*,\mu^k), \hat{\psi}_i(\mu^k,x^{k+1})\rangle. \end{aligned}$$

Then, using $\mathbf{Ass}_\Psi(\Psi_2)$, and the strong monotonicity of Ψ in $\mathbf{Ass}_\Psi(\Psi_1)$ that $\Psi(x,y) + \Psi(y,x) \le -\nu\|x-y\|^2, \forall x,y \in C$, we get

$$\begin{aligned}&\Psi(\mu^k,\mu_*^k) - \Psi(\mu^k,x^{k+1}) + \Psi(x^*,x^{k+1}) - \Psi(x^*,\mu_*^k)\\&\quad\le -\nu\|\mu^k - x^*\|^2 + \Upsilon\|\mu^k - x^*\|\|\mu^k - x^{k+1} - x^* + \mu_*^k\|.\end{aligned} \tag{15}$$

Combining (14) and (15), we get

$$\begin{aligned}0 &\le (1-2s_k\nu)\|\mu^k - x^*\|^2 - (\|x^{k+1} - \mu^k - \mu_*^k + x^*\| - s_k\Upsilon\|\mu^k - x^*\|)^2\\&\quad + s_k^2\Upsilon^2\|\mu^k - x^*\|^2 - \|x^{k+1} - \mu_*^k\|^2 \le (1-2s_k\nu + s_k^2\Upsilon^2)\|\mu^k - x^*\|^2 - \|x^{k+1} - \mu_*^k\|^2.\end{aligned}$$

From $0 < \lambda < \min\{\nu, \Upsilon\}$ and $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu-2\lambda}{\Upsilon^2-\lambda^2}\}$, it follows that $0 \le \eta_k = \sqrt{1-2s_k\nu + s_k^2\Upsilon^2} < 1 - \lambda s_k$. This ensures the desired result.
Step 4. We show that the sequence $\{x^k\}$ is bounded. Indeed, putting $X := C, Y := [0,1], \mathcal{G}(s) := C, \forall s \in Y$, $s := s_k, W(x,s) := -s\Psi(x^*,x) - \frac{1}{2}\|x-x^*\|^2, \forall (x,s) \in X \times Y$ we have $M(s_k) = \operatorname{argmin}\{s_k\Psi(x^*,x) + \frac{1}{2}\|x - x^*\|^2 : x \in C\} = \{\mu_*^k\}$. Note that M is continuous and $\lim_{k\to\infty}\mu_*^k = x^*$. Since Ψ is continuous on C, we get $\lim_{k\to\infty}\Psi(x^*,\mu_*^k) = \Psi(x^*,x^*) = 0$. In terms of $\mathbf{Ass}_\Psi(\Psi_3)$, there exists a constant $\hat{M}(x^*) > 0$ such that $|\Psi(x^*,\mu_*^k)| \le \hat{M}(x^*)\|\mu_*^k - x^*\|, \forall k \ge 1$. Putting $x = x^*$ in (3.9) and using $\Psi(x^*,x^*) = 0$, we get $-s_k\Psi(x^*,\mu_*^k) + \langle \mu_*^k - x^*, x^* - \mu_*^k\rangle \ge 0$, which hence yields $\|\mu_*^k - x^*\|^2 \le s_k[-\Psi(x^*,\mu_*^k)] \le s_k\hat{M}(x^*)\|\mu_*^k - x^*\|, \forall k \ge 1$. This immediately implies that $\|\mu_*^k - x^*\| \le s_k\hat{M}(x^*) \quad \forall k \ge 1$. Also, according to Lemma 2.3 we know that $I - \lambda_1F_1$ and $I - \lambda_2F_2$ are nonexpansive mappings, where $\lambda_1 \in (0,2\alpha)$ and $\lambda_2 \in (0,2\beta)$. Moreover, by Lemma 2.4, we know that G is nonexpansive. We write $y^* = J_{\lambda_2}^{B_2}(I - \lambda_2F_2)x^*$. Then, by Proposition 1.1, we get $x^* = J_{\lambda_1}^{B_1}(I - \lambda_1F_1)y^* = Gx^*$. So it follows that

$$\|p^k - x^*\| = \|G\mu^k - x^*\| \le \|\mu^k - x^*\|. \tag{16}$$

Also, in terms of $\sup_{k\ge1}\frac{\varepsilon_k}{s_k} < \infty$ and $x^k - x^{k+1} \to 0$, we know that $\frac{\varepsilon_k}{s_k}\|x^k - x^{k-1}\| \to 0$ as $k \to \infty$, which hence implies that there exists a constant $\widetilde{M}_0 > 0$ such that $\frac{\varepsilon_k}{s_k}\|x^k - x^{k-1}\| \le \widetilde{M}_0$, $\forall k \ge 1$. Therefore,

$$\begin{aligned}\|\tau^k - x^*\| &= \|x^k - x^* + \varepsilon_k(x^k - x^{k-1})\| \le \|x^k - x^*\| + \varepsilon_k\|x^k - x^{k-1}\|\\&= \|x^k - x^*\| + s_k\cdot\frac{\varepsilon_k}{s_k}\|x^k - x^{k-1}\| \le \|x^k - x^*\| + \widetilde{M}_0 s_k.\end{aligned} \tag{17}$$

Utilizing the result in Step 1, from (17) we get

$$\|z^k - x^*\| \le \|\tau^k - x^*\| \le \|x^k - x^*\| + \widetilde{M}_0 s_k,\ \forall k \ge 1. \tag{18}$$

Since each T_k is ζ-strictly pseudocontractive, G is a nonexpansive mapping and T is asymptotically nonexpansive, we deduce from $(\gamma_k + \delta_k)\zeta \le \gamma_k$ and Lemma 2.2 that

$$\begin{aligned}\|\mu^k - x^*\|^2 &\le \beta_k(1+\theta_k)^2\|z^k - x^*\|^2 + (1-\beta_k)\|G\mu^k - x^*\|^2\\&\quad - \beta_k(1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(T^kz^k - G\mu^k) + \frac{\delta_k}{1-\beta_k}(T^kz^k - T_kG\mu^k)\|^2,\end{aligned}$$

which together with (18) yields

$$\begin{aligned}\|\mu^k - x^*\|^2 &\le (1+\theta_k)^2\|z^k - x^*\|^2 - (1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(T^kz^k - G\mu^k)\\&\quad + \frac{\delta_k}{1-\beta_k}(T^kz^k - T_kG\mu^k)\|^2 \le (1+\theta_k)^2(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2.\end{aligned} \tag{19}$$

Consequently,

$$\begin{aligned}\|x^{k+1}-x^*\| &\le (1-\frac{1}{2}\lambda s_k)\|x^k-x^*\| + s_k[\widetilde{M}_0+\hat{M}(x^*)] \\ &\le \max\{\|x^k-x^*\|, \frac{2[\widetilde{M}_0+\hat{M}(x^*)]}{\lambda}\}.\end{aligned} \tag{20}$$

By induction, we get $\|x^k-x^*\| \le \max\{\|x^1-x^*\|, \frac{2[\widetilde{M}_0+\hat{M}(x^*)]}{\lambda}\}, \forall k \ge 1$. Thus, $\{x^k\}$ is bounded, and so are the sequences $\{p^k\},\{\tau^k\},\{\mu^k\},\{v^k\},\{y^k\},\{z^k\}$.

Step 5. We show that if $x^{k_i} \rightharpoonup \hat{x}$, $\tau^{k_i}-x^{k_i} \to 0$ and $\tau^{k_i}-y^{k_i} \to 0$ for $\{k_i\} \subset \{k\}$, then $\hat{x} \in \mathrm{Sol}(C,\Phi)$. Indeed, noticing $\tau^{k_i}-x^{k_i} \to 0$ and $\tau^{k_i}-y^{k_i} \to 0$, we get

$$\|x^{k_i}-y^{k_i}\| \le \|x^{k_i}-\tau^{k_i}\| + \|\tau^{k_i}-y^{k_i}\| \to 0 \quad (i \to \infty). \tag{21}$$

So it follows from $x^{k_i} \rightharpoonup \hat{x}$ that $\tau^{k_i} \rightharpoonup \hat{x}$ and $y^{k_i} \rightharpoonup \hat{x}$. Since $\{y^k\} \subset C$, $y^{k_i} \rightharpoonup \hat{x}$ and C is weakly closed, we know that $\hat{x} \in C$. By (3.3), we have $\alpha_{k_i}\Phi(\tau^{k_i},x) \ge \alpha_{k_i}\Phi(\tau^{k_i},y^{k_i}) + \langle y^{k_i}-\tau^{k_i}, y^{k_i}-x\rangle, \forall x \in C$. Taking the limit as $i \to \infty$ and using the assumptions that $\lim_{k\to\infty}\alpha_k = \tilde{\alpha} > 0$, $\Phi(\hat{x},\hat{x}) = 0$, $\{y^{k_i}\}$ is bounded and Φ is weakly continuous, we obtain that $\tilde{\alpha}\Phi(\hat{x},x) \ge 0\ \forall x \in C$. This implies that $\hat{x} \in \mathrm{sol}(C,\Phi)$.

Step 6. We claim that $x^k \to x^*$, a unique solution of the MBEP with the GSVI and CFPP constraints. Indeed, set $\mathit{\Gamma}_k = \|x^k-x^*\|^2$. By the results in Steps 1 and 2 we deduce from (18) and (19) that

$$\begin{aligned}\|x^{k+1}-x^*\|^2 \le{}& (\|x^k-x^*\| + \widetilde{M}_0 s_k)^2 + \theta_k\widetilde{M} - (1+\theta_k)^2[(1-2\alpha_k c_1)\|y^k-\tau^k\|^2 \\ &+ (1-2\alpha_k c_2)\|z^k-y^k\|^2] - (1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(T^k z^k - G\mu^k) \\ &+ \frac{\delta_k}{1-\beta_k}(T^k z^k - T_k G\mu^k)\|^2 - \|x^{k+1}-\mu^k\|^2 + s_k K,\end{aligned} \tag{22}$$

where $\sup_{k\ge1}\{(2+\theta_k)(\|x^k-x^*\|+\widetilde{M}_0 s_k)^2\} \le \widetilde{M}$ and $\sup_{k\ge1}\{2|\Psi(\mu^k,x^*)-\Psi(\mu^k,x^{k+1})|\} \le K$ for some $\widetilde{M}, K > 0$.

Finally, we show the convergence of $\{\mathit{\Gamma}_k\}$ to zero by the following two cases: Case 1. Suppose that there exists an integer $k_0 \ge 1$ such that $\{\mathit{\Gamma}_k\}$ is non-increasing. Then the limit $\lim_{k\to\infty}\mathit{\Gamma}_k = \hbar < +\infty$ and $\mathit{\Gamma}_k - \mathit{\Gamma}_{k+1} \to 0 \quad (k \to \infty)$. From (22), we get

$$\begin{aligned}&(1+\theta_k)^2[(1-2\alpha_k c_1)\|y^k-\tau^k\|^2 + (1-2\alpha_k c_2)\|z^k-y^k\|^2] \\ &+ (1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(T^k z^k - G\mu^k) + \frac{\delta_k}{1-\beta_k}(T^k z^k - T_k G\mu^k)\|^2 + \|x^{k+1}-\mu^k\|^2 \\ &\le \mathit{\Gamma}_k - \mathit{\Gamma}_{k+1} + \widetilde{M}_0 s_k(2\sqrt{\mathit{\Gamma}_k} + \widetilde{M}_0 s_k) + \theta_k\widetilde{M} + s_k K.\end{aligned} \tag{23}$$

Since $s_k \to 0$, $\theta_k \to 0$, $\mathit{\Gamma}_k - \mathit{\Gamma}_{k+1} \to 0$ and $0 < \liminf_{k\to\infty}(1-\beta_k)$, we obtain from $\{\alpha_k\} \subset (\underline{\alpha},\overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1},\frac{1}{2c_2}\})$ that

$$\lim_{k\to\infty}\|\frac{\gamma_k}{1-\beta_k}(T^k z^k - G\mu^k) + \frac{\delta_k}{1-\beta_k}(T^k z^k - T_k G\mu^k)\| = 0, \tag{24}$$

and

$$\lim_{k\to\infty}\|y^k-\tau^k\| = \lim_{k\to\infty}\|z^k-y^k\| = \lim_{k\to\infty}\|x^{k+1}-\mu^k\| = 0. \tag{25}$$

We now show that $\|\mu^k-p^k\| \to 0$ as $k \to \infty$. Indeed, we set $y^* = J_{\lambda_2}^{B_2}(x^*-\lambda_2F_2x^*)$. Note that $v^k = J_{\lambda_2}^{B_2}(\mu^k-\lambda_2F_2\mu^k)$ and $p^k = J_{\lambda_1}^{B_1}(v^k-\lambda_1F_1v^k)$. Then $p^k = G\mu^k$. By Lemma 2.3 we have

$$\|v^k-y^*\|^2 \le \|\mu^k-x^*\|^2 - \lambda_2(2\beta-\lambda_2)\|F_2\mu^k - F_2x^*\|^2, \tag{26}$$

and

$$\|p^k - x^*\|^2 \le \|v^k - y^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2. \tag{27}$$

Also, Lemma 2.2 together with (18) guarantees that $\|\mu^k - x^*\|^2 \le \frac{\beta_k}{2}[(1+\theta_k)^2(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \|\mu^k - x^*\|^2 - \|T^k z^k - \mu^k\|^2] + (1-\beta_k)\|\mu^k - x^*\|^2$ which hence leads to

$$\|\mu^k - x^*\|^2 \le (1+\theta_k)^2(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 - \|T^k z^k - \mu^k\|^2. \tag{28}$$

Substituting (26) for (27), by (28) we get

$$\begin{aligned}\|p^k - x^*\|^2 &\le \|\mu^k - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|F_2\mu^k - F_2 x^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2 \\ &\le (1+\theta_k)^2(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 - \|T^k z^k - \mu^k\|^2 - \lambda_2(2\beta - \lambda_2)\|F_2\mu^k - F_2 x^*\|^2 \\ &\quad - \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2.\end{aligned}$$

Moreover, substituting the last inequality for (22), from (18) we get

$$\begin{aligned}\|x^{k+1} - x^*\|^2 \le \theta_k \widetilde{M} - (1-\beta_k)[\|T^k z^k - \mu^k\|^2 + \lambda_2(2\beta - \lambda_2)\|F_2\mu^k - F_2 x^*\|^2 \\ + (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2] + s_k K,\end{aligned}$$

which immediately leads to

$$\begin{aligned}&(1-\beta_k)[\|T^k z^k - \mu^k\|^2 + \lambda_2(2\beta - \lambda_2)\|F_2\mu^k - F_2 x^*\|^2 + \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2] \\ &\le \Gamma_k - \Gamma_{k+1} + \widetilde{M}_0 s_k(2\sqrt{\Gamma_k} + \widetilde{M}_0 s_k) + \theta_k \widetilde{M} + s_k K.\end{aligned}$$

Since $\lambda_1 \in (0, 2\alpha)$, $\lambda_2 \in (0, 2\beta)$, $s_k \to 0$, $\theta_k \to 0$, $\Gamma_k - \Gamma_{k+1} \to 0$ and $\liminf_{k\to\infty}(1-\beta_k) > 0$, we get

$$\lim_{k\to\infty} \|T^k z^k - \mu^k\| = \lim_{k\to\infty} \|F_2\mu^k - F_2 x^*\| = \lim_{k\to\infty} \|F_1 v^k - F_1 y^*\| = 0. \tag{29}$$

On the other hand, observe that

$$\begin{aligned}&\|p^k - x^*\|^2 \le \langle v^k - y^*, p^k - x^*\rangle + \lambda_1\langle F_1 y^* - F_1 v^k, p^k - x^*\rangle \\ &\le \tfrac{1}{2}[\|v^k - y^*\|^2 + \|p^k - x^*\|^2 - \|v^k - p^k + x^* - y^*\|^2] + \lambda_1\|F_1 y^* - F_1 v^k\|\|p^k - x^*\|.\end{aligned}$$

This ensures that

$$\|p^k - x^*\|^2 \le \|v^k - y^*\|^2 - \|v^k - p^k + x^* - y^*\|^2 + 2\lambda_1\|F_1 y^* - F_1 v^k\|\|p^k - x^*\|. \tag{30}$$

Similarly, we get

$$\|v^k - y^*\|^2 \le \|\mu^k - x^*\|^2 - \|\mu^k - v^k + y^* - x^*\|^2 + 2\lambda_2\|F_2 x^* - F_2\mu^k\|\|v^k - y^*\|. \tag{31}$$

Combining (30) and (31), by (28) we have

$$\begin{aligned}\|p^k - x^*\|^2 \le (1+\theta_k)^2(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 - \|\mu^k - v^k + y^* - x^*\|^2 - \|v^k - p^k \\ + x^* - y^*\|^2 + 2\lambda_1\|F_1 y^* - F_1 v^k\|\|p^k - x^*\| + 2\lambda_2\|F_2 x^* - F_2\mu^k\|\|v^k - y^*\|.\end{aligned} \tag{32}$$

Substituting (32) for (22), from (18) we get

$$\begin{aligned}&\|x^{k+1} - x^*\|^2 \le (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \theta_k \widetilde{M} - (1-\beta_k)[\|\mu^k - v^k + y^* - x^*\|^2 + s_k K \\ &+ \|v^k - p^k + x^* - y^*\|^2] + 2\lambda_1\|F_1 y^* - F_1 v^k\|\|p^k - x^*\| + 2\lambda_2\|F_2 x^* - F_2\mu^k\|\|v^k - y^*\|.\end{aligned}$$

This immediately leads to

$$\begin{aligned}&(1-\beta_k)[\|\mu^k - v^k + y^* - x^*\|^2 + \|v^k - p^k + x^* - y^*\|^2] \le \Gamma_k - \Gamma_{k+1} + \theta_k \widetilde{M} + s_k K \\ &+ \widetilde{M}_0 s_k(2\sqrt{\Gamma_k} + \widetilde{M}_0 s_k) + 2\lambda_1\|F_1 y^* - F_1 v^k\|\|p^k - x^*\| + 2\lambda_2\|F_2 x^* - F_2\mu^k\|\|v^k - y^*\|.\end{aligned}$$

Since $s_k \to 0$, $\theta_k \to 0$, $\Gamma_k - \Gamma_{k+1} \to 0$ and $\liminf_{k\to\infty}(1-\beta_k) > 0$, we deduce from (29) that $\lim_{k\to\infty}\|\mu^k - v^k + y^* - x^*\| = 0$ and $\lim_{k\to\infty}\|v^k - p^k + x^* - y^*\| = 0$. Thus,

$$\|\mu^k - G\mu^k\| = \|\mu^k - p^k\| \le \|\mu^k - v^k + y^* - x^*\| + \|v^k - p^k + x^* - y^*\| \to 0 (k \to \infty). \tag{33}$$

Noticing $\tau^k = x^k + \varepsilon_k(x^k - x^{k-1})$, we deduce that $\|\tau^k - x^k\| = \varepsilon_k\|x^k - x^{k-1}\| \to 0(k \to \infty)$. Also, note that $0 = \beta_k(T^k z^k - \mu^k) + \gamma_k(p^k - \mu^k) + \delta_k(T_k p^k - \mu^k)$. Since $\liminf_{k\to\infty} \delta_k > 0$, from (29) and (33) we obtain that $\|T_k p^k - \mu^k\| \le \frac{1}{\delta_k}(\|T^k z^k - \mu^k\| + \|p^k - \mu^k\|) \to 0(k \to \infty)$ and hence $\|T_k\mu^k - \mu^k\| \le \frac{1+\zeta}{1-\zeta}\|\mu^k - p^k\| + \|T_k p^k - \mu^k\| \to 0(k \to \infty)$. Therefore,

$$\lim_{k\to\infty} \|\tau^k - x^k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T_k\mu^k - \mu^k\| = 0. \tag{34}$$

Using (25) and the assumption $x^k - x^{k+1} \to 0$, we get

$$\|x^k - \mu^k\| \le \|x^k - x^{k+1}\| + \|x^{k+1} - \mu^k\| \to 0,\ k \to \infty, \tag{35}$$

and

$$\|z^k - \tau^k\| \le \|z^k - y^k\| + \|y^k - \tau^k\| \to 0,\ k \to \infty. \tag{36}$$

Combining (33) and (35), we have

$$\|x^k - Gx^k\| \le 2\|x^k - \mu^k\| + \|\mu^k - G\mu^k\| \to 0, k \to \infty. \tag{37}$$

We claim that $\|T_k x^k - x^k\| \to 0$ and $\|Tx^k - x^k\| \to 0$ as $k \to \infty$. In fact, from (34) and (35) we deduce that

$$\|T_k x^k - x^k\| \le \frac{2}{1-\zeta}\|x^k - \mu^k\| + \|T_k\mu^k - \mu^k\| \to 0 \quad (k \to \infty). \tag{38}$$

Combining (34) and (36), we have

$$\|x^k - z^k\| \le \|x^k - \tau^k\| + \|\tau^k - z^k\| \to 0, k \to \infty. \tag{39}$$

Using (29), (35) and (39), we infer from the asymptotical nonexpansivity of T that

$$\|x^k - T^k x^k\| \le \|x^k - \mu^k\| + \|\mu^k - T^k z^k\| + (1 + \theta_k)\|z^k - x^k\| \to 0 \quad (k \to \infty). \tag{40}$$

This together with the assumption $\|T^k x^k - T^{k+1} x^k\| \to 0$, implies that

$$\|x^k - Tx^k\| \le (2 + \theta_1)\|x^k - T^k x^k\| + \|T^k x^k - T^{k+1} x^k\| \to 0 \quad (k \to \infty). \tag{41}$$

Next we show that $\lim_{k\to\infty} \|x^k - x^*\| = 0$. Indeed, since the sequences $\{\mu^k\}$ and $\{x^k\}$ are bounded, we know that there exists a subsequence $\{\mu^{k_i}\}$ of $\{\mu^k\}$ converging weakly to $\hat{x} \in C$ and satisfying the equality

$$\liminf_{k\to\infty}[\Psi(x^*, \mu^k) + \Psi(\mu^k, x^{k+1})] = \lim_{i\to\infty}[\Psi(x^*, \mu^{k_i}) + \Psi(\mu^{k_i}, x^{k_i+1})]. \tag{42}$$

From (25) and (35) it follows that $x^{k_i} \rightharpoonup \hat{x}$ and $x^{k_i+1} \rightharpoonup \hat{x}$. Then, by the result in Step 5, we deduce that $\hat{x} \in \mathrm{Sol}(C, \Phi)$.

It is clear from (41) that $x^{k_i} - Tx^{k_i} \to 0$. Note that Lemma 2.6 guarantees the demiclosedness of $I - T$ at zero. So, we know that $\hat{x} \in \mathrm{Fix}(T)$. Also, note that Lemma 2.6 guarantees the demiclosedness of $I - G$ at zero. Hence, from $x^{k_i} \rightharpoonup \hat{x}$ and $x^k - Gx^k \to 0$ (due to (37)) it follows that $\hat{x} \in \mathrm{Fix}(G)$. We claim that $\hat{x} \in \bigcap_{j=1}^N \mathrm{Fix}(T_j)$. As a matter of fact, since $x^k - \mu^k \to 0$ and $\mu^{k_i} \rightharpoonup \hat{x}$, we get $x^{k_i} \rightharpoonup \hat{x}$. Without loss of generality, we may assume $l = k_i \mathrm{mod} N$ for all i. Since $x^k - x^{k+1} \to 0$, we have $x^{k_i+j} \rightharpoonup \hat{x}$ for all $j \ge 1$. Moreover, from (38) we deduce that

$$\|x^{k_i+j} - T_{l+j}x^{k_i+j}\| = \|x^{k_i+j} - T_{k_i+j}x^{k_i+j}\| \to 0. \tag{43}$$

Note that Lemma 2.1 guarantees the demiclosedness of $I - T_i$ at zero for $i = 1, ..., N$. So it follows that $\hat{x} \in \mathrm{Fix}(T_{l+j})$ for all j. This ensures that $\hat{x} \in \bigcap_{j=1}^N \mathrm{Fix}(T_j)$. Consequently, $\hat{x} \in \bigcap_{j=0}^N \mathrm{Fix}(T_j) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi) = \Omega$. In terms of (42), we have

$$\liminf_{k\to\infty}[\Psi(x^*, \mu^k) + \Psi(\mu^k, x^{k+1})] = \Psi(x^*, \hat{x}) \ge 0. \tag{44}$$

Since Ψ is ν-strongly monotone, we have

$$\limsup_{k\to\infty}[\Psi(x^*,\mu^k)+\Psi(\mu^k,x^*)]\le\limsup_{k\to\infty}(-\nu\|\mu^k-x^*\|^2)=-\nu\hbar. \tag{45}$$

Combining (44) and (45), we obtain

$$\begin{aligned}\limsup_{k\to\infty}[\Psi(\mu^k,x^*)-\Psi(\mu^k,x^{k+1})]&\le\limsup_{k\to\infty}[\Psi(\mu^k,x^*)+\Psi(x^*,\mu^k)]\\&\quad-\liminf_{k\to\infty}[\Psi(x^*,\mu^k)+\Psi(\mu^k,x^{k+1})]\le-\nu\hbar.\end{aligned} \tag{46}$$

We now claim that $\hbar=0$. On the contrary we assume $\hbar>0$. Without loss of generality we may assume that $\exists k_0\ge1$ s.t.

$$\Psi(\mu^k,x^*)-\Psi(\mu^k,x^{k+1})\le-\frac{\nu\hbar}{2},\ \forall k\ge k_0, \tag{47}$$

which together with (22), implies that for all $k\ge k_0$,

$$\begin{aligned}\|x^{k+1}-x^*\|^2&\le(1+\theta_k)^2\|\tau^k-x^*\|^2+2s_k[\Psi(\mu^k,x^*)-\Psi(\mu^k,x^{k+1})]\\&\le\|x^k-x^*\|^2+\varepsilon_k\|x^k-x^{k-1}\|\widetilde{M}_1+\theta_k\widetilde{M}-s_k\nu\hbar,\end{aligned} \tag{48}$$

for some $\widetilde{M}_1>0$. So it follows that for all $k\ge k_0$,

$$\Gamma_k-\Gamma_{k_0}\le\sum_{j=k_0}^{k-1}(\varepsilon_j\|x^j-x^{j-1}\|\widetilde{M}_1+\theta_j\widetilde{M})-\nu\hbar\sum_{j=k_0}^{k-1}s_j. \tag{49}$$

Taking the limit in (49) as $k\to\infty$, we get $-\infty<\hbar-\Gamma_{k_0}\le\lim_{k\to\infty}[\sum_{j=k_0}^{k-1}(\varepsilon_j\|x^j-x^{j-1}\|\widetilde{M}_1+\theta_j\widetilde{M})-\nu\hbar\sum_{j=k_0}^{k-1}s_j]=-\infty$. This reaches a contradiction. Therefore, $\lim_{k\to\infty}\Gamma_k=0$ and hence $\{x^k\}$ converges strongly to the unique solution x^* of the problem EP(Ω,Ψ).

Case 2. Suppose that $\exists\{\Gamma_{k_j}\}\subset\{\Gamma_k\}$ s.t. $\Gamma_{k_j}<\Gamma_{k_j+1}$ $\forall j\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau:\mathcal{N}\to\mathcal{N}$ by $\tau(k):=\max\{j\le k:\Gamma_j<\Gamma_{j+1}\}$. By Lemma 2.7, we get

$$\Gamma_{\tau(k)}\le\Gamma_{\tau(k)+1}\quad\text{and}\quad\Gamma_k\le\Gamma_{\tau(k)+1}. \tag{50}$$

Utilizing the same inferences as in (25) and (35), we can obtain that

$$\lim_{k\to\infty}\|x^{\tau(k)+1}-\mu^{\tau(k)}\|=\lim_{k\to\infty}\|\tau^{\tau(k)}-y^{\tau(k)}\|=\lim_{k\to\infty}\|y^{\tau(k)}-z^{\tau(k)}\|=0, \tag{51}$$

and

$$\lim_{k\to\infty}\|x^{\tau(k)}-\mu^{\tau(k)}\|=0. \tag{52}$$

Since $\{\mu^k\}$ is bounded, there exists a subsequence of $\{\mu^{\tau(k)}\}$ converging weakly to $\hat{x}$. Without loss of generality, we may assume that $\mu^{\tau(k)}\rightharpoonup\hat{x}$. Then, utilizing the same inferences as in Case 1, we can obtain that $\hat{x}\in\Omega=\bigcap_{i=0}^N\mathrm{Fix}(T_i)\cap\mathrm{Fix}(G)\cap\mathrm{Sol}(C,\Phi)$. From $\mu^{\tau(k)}\rightharpoonup\hat{x}$ and (51), we get $x^{\tau(k)+1}\rightharpoonup\hat{x}$. Using the condition $\{\alpha_k\}\subset(\underline{\alpha},\overline{\alpha})\subset(0,\min\{\frac{1}{2c_1},\frac{1}{2c_2}\})$, we have $1-2\alpha_{\tau(k)}c_1>0$ and $1-2\alpha_{\tau(k)}c_2>0$. So it follows from (22) that

$$\Psi(\mu^{\tau(k)},x^{\tau(k)+1})-\Psi(\mu^{\tau(k)},x^*)\le\frac{\varepsilon_{\tau(k)}}{s_{\tau(k)}}\|x^{\tau(k)}-x^{\tau(k)-1}\|\cdot\frac{\widetilde{M}_1}{2}+\frac{\theta_{\tau(k)}}{s_{\tau(k)}}\cdot\frac{\widetilde{M}}{2}. \tag{53}$$

Since Ψ is ν-strongly monotone on C, we get

$$\nu\|\mu^{\tau(k)}-x^*\|^2\le-\Psi(\mu^{\tau(k)},x^*)-\Psi(x^*,\mu^{\tau(k)}). \tag{54}$$

Combining (53) and (54), we deduce from $\sup_{k\ge1}\frac{\varepsilon_k}{s_k}<\infty$, $x^k-x^{k+1}\to0$, $\mathbf{Ass}_\Psi(\Psi_1)$ and $\hat{x}\in\Omega$ that

$$\nu\limsup_{k\to\infty}\|\mu^{\tau(k)}-x^*\|^2\le\limsup_{k\to\infty}[-\Psi(\mu^{\tau(k)},x^{\tau(k)+1})-\Psi(x^*,\mu^{\tau(k)})]\le0. \tag{55}$$

Hence, $\limsup_{k\to\infty} \|x^{\tau(k)} - x^*\|^2 \le 0$. Thus, we get $\lim_{k\to\infty} \|x^{\tau(k)} - x^*\|^2 = 0$. From (52), we get

$$\begin{aligned} \|x^{\tau(k)+1} - x^*\|^2 - \|x^{\tau(k)} - x^*\|^2 \le 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| \\ + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2 \to 0 \quad (k \to \infty). \end{aligned} \tag{56}$$

Owing to $\Gamma_k \le \Gamma_{\tau(k)+1}$, we get

$$\|x^k - x^*\|^2 \le \|x^{\tau(k)} - x^*\|^2 + 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2. \tag{57}$$

So it follows from (52) that $x^k \to x^*$ as $k \to \infty$. This completes the proof. □

Algorithm 3.1. *Let $x^0, x^1 \in C$ be arbitrary. The sequences $\{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0,1)$, and positive sequences $\{\alpha_k\}, \{\varepsilon_k\}, \{s_k\}$ satisfy conditions (H1)-(H5). Calculate x^{k+1} as follows: Step 1. Compute $\tau^k = x^k + \varepsilon_k(x^k - x^{k-1})$ and $y^k = \operatorname{argmin}\{\alpha_k \Phi(\tau^k, y) + \frac{1}{2}\|y - \tau^k\|^2 : y \in C\}$. Step 2. Choose $w^k \in \partial_2 \Phi(\tau^k, y^k)$. Compute $C_k = \{v \in \mathcal{H} : \langle \tau^k - \alpha_k w^k - y^k, v - y^k\rangle \le 0\}$ and $z^k = \operatorname{argmin}\{\alpha_k \Phi(y^k, z) + \frac{1}{2}\|z - \tau^k\|^2 : z \in C_k\}$. Step 3. Compute $\mu^k = \beta_k T^k p^k + \gamma_k \mu^k + \delta_k T_k \mu^k$, $v^k = J_{\lambda_2}^{B_2}(z^k - \lambda_2 F_2 z^k)$ and $p^k = J_{\lambda_1}^{B_1}(v^k - \lambda_1 F_1 v^k)$. Step 4. Compute $x^{k+1} = \operatorname{argmin}\{s_k \Psi(\mu^k, t) + \frac{1}{2}\|t - \mu^k\|^2 : t \in C\}$. Set $k := k+1$ and return to Step 1.*

Using the similar arguments to those in the proof of Theorem 3.1, we can obtain the following convergence theorem.

Theorem 3.2. *Suppose that $\{x^k\}$ is the sequence defined by Algorithm 3.1, such that $\|T^k x^k - T^{k+1} x^k\| \to 0$ as $k \to \infty$. Let the bifunctions Ψ, Φ satisfy the assumptions* $\mathbf{Ass}_\Phi$-$\mathbf{Ass}_\Psi$*, and assume that the conditions (H1)-(H5) hold. Then $x^k \to x^* \in \Omega \Leftrightarrow x^k - x^{k+1} \to 0$, where $x^* \in \Omega$ is a unique solution to the problem EP(Ω, Ψ).*

4. Concluding Remarks

In this article, we have suggested two new iterative algorithms based on the Mann implicit inertial subgradient extragradient method for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. The strong convergence results for the proposed algorithms to solve such a MBEP with the GSVI and CFPP constraints are established under some mild assumptions. Furthermore, in the proposed method, the second minimization problem over a closed convex set is replaced by the subgradient projection onto some constructible half-space, and a new approach for solving the GSVI and CFPP via Mann implicit iterations is provided.

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