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QUANTUM INTEGRAL INEQUALITIES ON THE PATTERN OF OSTROWSKI AND OSTROWSKI-GRÜSS TYPE INEQUALITIES

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In this paper, we establish some well known integral inequalities for quantum calculus. By applying q-integral and q-derivative formulas Ostrowski and Ostrowski-Grüss type inequalities for q-integrals are obtained. In particular cases, some interesting consequences are produced. Two versions of Hadamard type inequalities are given in premises of quantum calculus.

Keywords: Ostrowski inequality; Ostrowski-Grüss inequality; q-derivative; q-integral; Hadamard inequality.

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1. Introduction

Classical mathematical concepts and notions are core entities in the development of new and generalized theories. For instance concept of limit was initiated to determine the value of a function at a point where it is undetermined. Later on it was used to define the derivative of a function which leads to the mathematical modeling of real world problems in the form of initial and boundary value problems, dynamical systems, control systems. The concept of fractional order derivatives was initiated at the same time when the usual derivative was invented. In this modern age of technology and artificial intelligence mathematical notions have been extended and generalized in plenty of ways.

Fractional derivatives and integrals of various types have been invented, and are applied for formulating generalized theories and notations. In solving fractional differential equations one need new kinds of special functions such as Mittag-Leffler function, hypergeometric function etc. More generally speaking theory of fractional calculus is actually applicable not only in mathematics but also in biology, medicine, mechanics, control theory and many other disciplines.

The notions q-derivative and q-integral are the generalizations of usual derivative and integral, that have interesting applications engineering and sciences. Mathematical concepts linked with ordinary derivatives and integrals have been converted into q-calculus such as q-polynomials, q-Taylor formula, q-gamma function, q-beta function q-hypergeometric function, q-Laplace transform etc can be found in literature. For a detailed study and applications of q-calculus we refer the readers to [1, 2, 3, 4, 5, 6, 7]. The q-derivative is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{1}$$

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From equation (1) it can be found that

$$D_q f(qx) = \frac{f(q^2x) - f(x)}{(q-1)x} - D_q f(x). \tag{2}$$

The above formulas (1) and (2) will be used frequently in establishing results of this paper. The q-integral is defined by

$$\int_0^x f(t)d_qt = x(1-q)\sum_{i=0}^{\infty} q^i f(q^i x),$$

and

$$\int_a^x f(t)d_qt = \int_0^x f(t)d_qt - \int_0^a f(t)d_qt.$$

Since, $D_q\{f(t)g(t)\} = f(t)D_qg(t) + g(qt)D_qf(t)$ the following formula of q-integration by parts is important to establish inequalities of this paper, see [1, p. 74]

$$\int_a^b f(t)D_qg(t)d_qt = f(b)g(b) - f(a)g(a) - \int_a^b g(qt)D_qf(t)d_qt. \tag{3}$$

The q-gamma function of a non-negative integer n is defined by

$$\Gamma_q(n+1) = 1(1+q)(1+q+q^2)...(1+q+...+q^{n-1}). \tag{4}$$

Also, $\lim_{q\to 1^-} \Gamma_q(n+1) = \Gamma(n+1) = n!$, see [2].

Another definition of q-derivative on an interval $[a,b]$ is given as follows, see [8]:

Definition 1.1. *Let $u : I = [a,b] \to \mathbb{R}$ be a continuous function. For $0 < q < 1$ the q-derivative ${}_a\mathcal{D}_q u$ on I, is given by;*

$${}_a\mathcal{D}_q u(\xi) := \frac{u(q\xi + (1-q)a) - u(\xi)}{(q-1)(\xi - a)}, \ \xi \neq a, \ {}_a\mathcal{D}_q u(a) = \lim_{\xi\to a} {}_a\mathcal{D}_q u(\xi). \tag{5}$$

Definition 1.2. [8] *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the q-definite integral on $[a,b]$ is defined as*

$$\int_a^x f(t)d_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{6}$$

for $x \in [a,b]$, $a, b \in I$. If $c \in (a,x)$, then we have

$$\int_c^x f(t)d_qt = \int_a^x f(t)d_qt - \int_a^c f(t)d_qt. \tag{7}$$

All the above definitions and formulas will be utilized where needed, and we establish quantum versions of well known inequalities that exist in literature. Forthcoming results are worthy to mention for best understanding of the reason behind formulating results of this paper. Let we narrate historically some classical inequalities and their related improvements/refinements.

A convex function was defined in the start of twentieth century and satisfy a classical inequality, known as Hermite-Hadamard inequality and given as follows:

$$\Psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \Psi(\lambda)d\lambda \leq \frac{\Psi(u_1)+\Psi(u_2)}{2}, \tag{8}$$

provided Ψ is convex on $[u_1, u_2]$. This inequality is very commonly targeted by researchers for further possible investigations. For instance it is established in routine for new classes of functions as well as for different kinds of integrals including fractional integrals, quantum integrals, conformable integrals etc. It is also very common to find its refinements/generalizations, the Hermite-Hadamard inequality is also estimated in the form of error bounds. A well known inequality namely Ostrowski inequality also provides error bounds of inequality (8) by applying it at mid as well as boundary points of the defined interval. The Ostrowski inequality is stated as follows:

Theorem 1.1. *Let $\Psi : I \longrightarrow \mathbb{R}$, be a differentiable mapping in I^o, the interior of I and $u_1, u_2 \in I^o$, $u_1 < u_2$. If $|\Psi'(t)| \leq \mathcal{M}$ for all $t \in [u_1, u_2]$. Then for $\sigma \in [u_1, u_2]$ one can get:*

$$\left|\Psi(\sigma) - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \Psi(\lambda)d\lambda\right| \leq \left[\frac{1}{4} + \frac{(\sigma - \frac{u_1+u_2}{2})^2}{(u_2 - u_1)^2}\right](u_2 - u_1)\mathcal{M}. \tag{9}$$

By applying a well known Grüss inequality [11], an Ostrowski type inequality was derived in [12]. In literature it is called Ostrowski-Grüss inequality. In [13], the following Ostrowski-Grüss type inequality was established:

Theorem 1.2. *Let $\Psi : I \longrightarrow \mathbb{R}$, be a differentiable mapping in I^o, the interior of I and $u_1, u_2 \in I^o$, $u_1 < u_2$. If $\mathcal{M}_1 \leq \Psi'(\lambda) \leq \mathcal{M}_2$ for all $\lambda \in [u_1, u_2]$. Then for $\sigma \in [u_1, u_2]$ we have*

$$\left|\frac{1}{2}\Psi(\sigma) - \frac{(\sigma - u_2)\Psi(u_2) - (\sigma - u_1)\Psi(u_1)}{2(u_2 - u_1)} - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \Psi(\lambda)d\lambda\right| \tag{10}$$
$$\leq \frac{(\sigma - u_1)^2 + (u_2 - \sigma)^2}{4(u_2 - u_1)}(\mathcal{M}_2 - \mathcal{M}_1).$$

Our goal in this paper is, to investigate inequalities with conditions similar to the aforementioned results but in quantum calculus. In forthcoming section, we first derive Ostrowski inequality for q-integrals by setting condition on q-derivative to be bounded by an arbitrary function. Some consequent inequalities have direct link with recently published inequalities. An inequality similar to (10) for q-integrals is proved. We also find two inequalities applying condition on second order q-derivative to be bounded.

2. Quantum Integral Inequalities of Ostrowski Type and Linked Results

Theorem 2.1. *Let $\Psi : [u_1, u_2] \longrightarrow \mathbb{R}$ be q-differentiable and $|D_q\Psi(\lambda)| \leq \eta(\lambda)$, $\lambda \in [u_1, u_2]$, where η is q-integrable. Then for $\sigma \in [u_1, u_2]$, one can have*

$$\left|\Psi(\sigma) - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \Psi(q\lambda)d_q\lambda\right| \tag{11}$$
$$\leq \frac{1}{u_2 - u_1}\left((\sigma - u_1)\int_{u_1}^{\sigma} \eta(\lambda)d_q\lambda + (u_2 - \sigma)\int_{\sigma}^{u_2} \eta(\lambda)d_q\lambda\right).$$

Proof. Under given conditions it is easy to see that the following expression is non-negative:

$$\int_{u_1}^{\sigma}(\lambda - u_1)D_q(\Psi(\lambda))d_q\lambda + \int_{u_1}^{\sigma}(\lambda - u_1)\eta(\lambda)d_q\lambda,$$

where $\lambda \in [u_1, \sigma]$ and $\sigma \in [u_1, u_2]$. By using q-integration by parts, one can have the following inequality for q-integrals:

$$(\sigma - u_1)\Psi(\sigma) - \int_{u_1}^{\sigma} \Psi(q\lambda)d_q\lambda \geq -(\sigma - u_1)\int_{u_1}^{\sigma} \eta(\lambda)d_q\lambda. \tag{12}$$

On similar way, for $\lambda \in [\sigma, u_2]$, one can get the following inequality for q-integrals:

$$\int_{\sigma}^{u_2}(u_2 - \lambda)\eta(\lambda)d_q\lambda - \int_{\sigma}^{u_2}(u_2 - \lambda)D_q\Psi(\lambda)d_q\lambda \geq 0.$$

By using q-integration by parts, one can have the following inequality for q-integrals:

$$(u_2 - \sigma)\int_{\sigma}^{u_2} \eta(\lambda)d_q\lambda + (u_2 - \sigma)\Psi(\sigma) - \int_{\sigma}^{u_2} \Psi(q\lambda)d_q\lambda \geq 0. \tag{13}$$

After some simplifications, adding the inequalities (12) and (13), the following inequality for q-integrals hold:

$$\Psi(\sigma)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda \tag{14}$$
$$\geq -\frac{1}{u_2-u_1}\left((\sigma-u_1)\int_{u_1}^{\sigma}\eta(\lambda)d_q\lambda+(u_2-\sigma)\int_{\sigma}^{u_2}\eta(\lambda)d_q\lambda\right).$$

Also, for $\lambda\in[u_1,u_2]$ the terms $(\lambda-u_1)(\eta(\lambda)-D_q(\Psi(\lambda)))$ and $(u_2-\lambda)(\eta(\lambda)+D_q(\Psi(\lambda)))$, are non-negative. Hence, for q-integrals the following inequality holds:

$$\int_{u_1}^{\sigma}(\lambda-u_1)(\eta(\lambda)-D_q(\Psi(\lambda)))d_q\lambda+\int_{\sigma}^{u_2}(u_2-\lambda)(\eta(\lambda)+D_q(\Psi(\lambda)))d_q\lambda\geq 0. \tag{15}$$

Making q-integration by parts one can get the following q-integral inequality:

$$\Psi(\sigma)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda \tag{16}$$
$$\leq \frac{1}{u_2-u_1}\left((\sigma-u_1)\int_{u_1}^{\sigma}\eta(\lambda)d_q\lambda+(u_2-\sigma)\int_{\sigma}^{u_2}\eta(\lambda)d_q\lambda\right).$$

Inequalities (14) and (16), provide the required inequality. □

Some examples of function η are considered to get the forthcoming results.

Remark 2.1. *By setting specific values of function η, one can obtain Ostrowski type inequalities of various forms. We consider two examples as follows:*
1. *Let*

$$\eta(\lambda)=\begin{cases}\frac{||D_q\Psi||}{\sigma-u_1}(\sigma-\lambda), & \text{if } u_1\leq\lambda\leq\sigma\\ \frac{||D_q\Psi||}{u_2-\sigma}(\lambda-\sigma), & \text{if } \sigma<\lambda\leq u_2.\end{cases} \tag{17}$$

Then inequality (11) *takes the following form*

$$\left|\Psi(\sigma)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda\right|$$
$$\leq\frac{1}{u_2-u_1}\left(\int_{u_1}^{\sigma}(\sigma-\lambda)d_q\lambda+\int_{\sigma}^{u_2}(\lambda-\sigma)d_q\lambda\right).$$

By calculating the q-integrals appearing in right hand side we get the following inequality:

$$\left|\Psi(\sigma)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda\right|$$
$$\leq ||D_q\Psi||(u_2-u_1)\left(\frac{2q}{1+q}\left(\frac{\sigma-\frac{(3q-1)a+(1+q)b}{4q}}{u_2-u_1}\right)^2+\frac{-q^2+6q-1}{8q(1+q)}\right).$$

The above inequality is exclusively proved in [8].
2. *Let $\eta(\lambda)=(\lambda-u_1)^\beta$, $\beta\in\mathbb{R}-\{-1\}$. Then we have*

$$\int_{u_1}^{\sigma}(\lambda-u_1)^\beta d_q\lambda=\frac{(\sigma-u_1)^{1+\beta}(1-q)}{(1-q^{1+\beta})}, \tag{18}$$

and

$$\int_{\sigma}^{u_2}(\lambda-u_1)^{\beta}d_q\lambda=\int_{u_1}^{u_2}(\lambda-u_1)^{\beta}d_q\lambda-\int_{u_1}^{\sigma}(\lambda-u_1)^{\beta}d_q\lambda \tag{19}$$
$$=\frac{(u_2-u_1)^{1+\beta}(1-q)}{(1-q^{1+\beta})}-\frac{(\sigma-u_1)^{1+\beta}(1-q)}{(1-q^{1+\beta})}$$
$$=\frac{1-q}{(1-q^{1+\beta})}\left((u_2-u_1)^{1+\beta}-(\sigma-u_1)^{1+\beta}\right),$$

see [8, Lemma 2.2]*. The inequality* (11)*, after putting values of q-integrals from* (18) *and* (19) *takes the following form:*

$$\left|\Psi(\sigma)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda\right|\leq\frac{1-q}{(u_2-u_1)(1-q^{1+\beta})}$$
$$\times\left((\sigma-u_1)^{2+\beta}+(u_2-\sigma)\left((u_2-u_1)^{1+\beta}-(\sigma-u_1)^{1+\beta}\right)\right).$$

Corollary 2.1. *If η is increasing along with conditions of Theorem 2.1, then we have*

$$\left|\Psi(\sigma)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda\right| \tag{20}$$
$$\leq\frac{1}{u_2-u_1}\left((\sigma-u_1)^2\eta(\sigma)+(u_2-\sigma)^2\eta(u_2)\right),\sigma\in[u_1,u_2].$$

If we set $\sigma=\frac{u_1+u_2}{2}$ in (20)*, it appears as follows:*

$$\left|\Psi\left(\frac{u_1+u_2}{2}\right)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda\right|\leq\frac{u_2-u_1}{4}\left(\eta\left(\frac{u_1+u_2}{2}\right)+\eta(u_2)\right).$$

The second main result is q-Ostrowski-Grüss type inequality stated and proved as follows:

Theorem 2.2. *Let $\Psi:[u_1,u_2]\longrightarrow\mathbb{R}$ be q-differentiable and $\zeta(t)\leq D_q\Psi(t)\leq\eta(t)$, $t\in[u_1,u_2]$, where ζ,η are q-integrable. Then for $\sigma\in[u_1,u_2]$, we have:*

$$\left|\frac{1}{2}\Psi(\sigma)-\frac{(\sigma-u_2)\Psi(u_2)-(\sigma-u_1)\Psi(u_1)}{2(u_2-u_1)}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda\right| \tag{21}$$
$$\leq\frac{1}{2(u_2-u_1)}\left[(\sigma-u_1)\int_{u_1}^{\sigma}\eta(\lambda)d_q\lambda-\int_{u_1}^{\sigma}(\sigma-\lambda)\zeta(\lambda)d_q\lambda+(u_2-\sigma)\int_{\sigma}^{u_2}\eta(\lambda)d_q\lambda\right.$$
$$\left.-\int_{\sigma}^{u_2}(u_2-\lambda)\zeta(\lambda)d_q\lambda\right].$$

Proof. Under given conditions it is easy to see that the following expression is non-negative:

$$\int_{u_1}^{\sigma}(\sigma-\lambda)(D_q\Psi(\lambda)-\zeta(\lambda))d_q\lambda+\int_{u_1}^{\sigma}(\lambda-u_1)(\eta(\lambda)-D_q\Psi(\lambda))d_q\lambda.$$

Hence we can obtain the following inequality

$$\int_{u_1}^{\sigma}(\sigma-\lambda)D_q\Psi(\lambda)d_q\lambda-\int_{u_1}^{\sigma}(\lambda-u_1)D_q\Psi(\lambda)d_q\lambda \tag{22}$$
$$\geq\int_{u_1}^{\sigma}(\sigma-\lambda)\zeta(\lambda)d_q\lambda-\int_{u_1}^{\sigma}(\lambda-u_1)\eta(\lambda)d_q\lambda.$$

Using formula (3) for q-integration by parts we have

$$(\sigma-u_1)\big(\Psi(u_1)+\Psi(\sigma)\big)-2\int_{u_1}^{\sigma}\Psi(q\lambda)d_q\lambda\leq(\sigma-u_1)\int_{u_1}^{\sigma}\eta(\lambda)d_q\lambda-\int_{u_1}^{\sigma}(\sigma-\lambda)\zeta(\lambda)d_q\lambda. \tag{23}$$

Under given conditions we also have the following q-integral inequality:

$$\int_{\sigma}^{u_2}(\lambda-\sigma)(\eta(\lambda)-D_q\Psi(\lambda))d_q\lambda+\int_{\sigma}^{u_2}(u_2-\lambda)(D_q\Psi(\lambda)-\zeta(\lambda))d_q\lambda\geq 0.$$

Which can be rewritten as follows:

$$\begin{aligned}&\int_{\sigma}^{u_2}(u_2-\lambda)D_q\Psi(\lambda)d_q\lambda-\int_{\sigma}^{u_2}(\lambda-\sigma)D_q\Psi(\lambda)d_q\lambda \\ &\geq\int_{\sigma}^{u_2}(u_2-\lambda)\zeta(\lambda)d_q\lambda-\int_{\sigma}^{u_2}(\lambda-\sigma)\eta(\lambda)d_q\lambda.\end{aligned}\tag{24}$$

Doing q-integration by parts using formula (3), and also by using $\lambda-\sigma\leq u_2-\sigma$ for $\lambda\in[\sigma,u_2]$, we have

$$\begin{aligned}&(u_2-\sigma)\big(\Psi(u_2)+\Psi(\sigma)\big)-2\int_{\sigma}^{u_2}\Psi(q\lambda)d_q\lambda \\ &\leq(u_2-\sigma)\int_{\sigma}^{u_2}\eta(\lambda)d_q\lambda-\int_{\sigma}^{u_2}\zeta(\lambda)(u_2-\lambda)d_q\lambda.\end{aligned}\tag{25}$$

From inequalities (23) and (25), one can get

$$\begin{aligned}&\frac{1}{2}\Psi(\sigma)-\frac{(\sigma-u_2)\Psi(u_2)-(\sigma-u_1)\Psi(u_1)}{2(u_2-u_1)}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda \\ &\leq\frac{1}{2(u_2-u_1)}\Bigg[(\sigma-u_1)\int_{u_1}^{\sigma}\eta(\lambda)d_q\lambda-\int_{u_1}^{\sigma}(\sigma-\lambda)\zeta(\lambda)d_q\lambda+(u_2-\sigma)\int_{\sigma}^{u_2}\eta(\lambda)d_q\lambda \\ &-\int_{\sigma}^{u_2}(u_2-\lambda)\zeta(\lambda)d_q\lambda\Bigg].\end{aligned}\tag{26}$$

Now, from given condition, one can obtain the forthcoming q-integral inequalities:

$$\int_{u_1}^{\sigma}(\lambda-u_1)(D_q\Psi(\lambda)-\zeta(\lambda))d_q\lambda+\int_{u_1}^{\sigma}(\sigma-\lambda)(\eta(\lambda)-D_q\Psi(\lambda))d_q\lambda\geq 0.\tag{27}$$

$$\int_{\sigma}^{u_2}(u_2-\lambda)(\eta(\lambda)-D_q\Psi(\lambda))d_q\lambda+\int_{\sigma}^{u_2}(\lambda-\sigma)(D_q\Psi(\lambda)-\zeta(\lambda))d_q\lambda\geq 0.\tag{28}$$

Applying q-integration by parts using formula (3) for aforementioned inequalities, and adding consequent inequalities, we get

$$\begin{aligned}&\frac{1}{2}\Psi(\sigma)-\frac{(\sigma-u_2)\Psi(u_2)-(\sigma-u_1)\Psi(u_1)}{2(u_2-u_1)}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda \\ &\geq-\frac{1}{2(u_2-u_1)}\Bigg[(\sigma-u_1)\int_{u_1}^{\sigma}\eta(\lambda)d_q\lambda-\int_{u_1}^{\sigma}(\sigma-\lambda)\zeta(\lambda)d_q\lambda+(u_2-\sigma)\int_{\sigma}^{u_2}\eta(\lambda)d_q\lambda \\ &-\int_{\sigma}^{u_2}(u_2-\lambda)\zeta(\lambda)d_q\lambda\Bigg].\end{aligned}\tag{29}$$

The required inequality (21) is obtained from inequalities (26) and (29). □

Corollary 2.2. *If η and ζ are increasing along with conditions of Theorem 2.2, then we have*

$$\begin{aligned}&\left|\frac{1}{2}\Psi(\sigma)-\frac{(\sigma-u_2)\Psi(u_2)-(\sigma-u_1)\Psi(u_1)}{2(u_2-u_1)}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Psi(q\lambda)d_q\lambda\right| \\ &\leq\frac{1}{2(u_2-u_1)}\left[(\sigma-u_1)^2\left(\eta(\sigma)-\frac{\zeta(u_1)}{2}\right)+(u_2-\sigma)^2\left(\eta(u_2)-\frac{\zeta(\sigma)}{2}\right)\right].\end{aligned}\tag{30}$$

In the following theorem we give a result for twice q-differentiable bounded functions.

Theorem 2.3. *Let $\Psi : [u_1, u_2] \longrightarrow \mathbb{R}$ have second order q-derivative to be bounded i.e $\zeta(\lambda) \le D_q^2\Psi(\lambda) \le \eta(\lambda)$ for all $\lambda \in [u_1, u_2]$, where $\zeta, \eta \in L[u_1, u_2]$. Then for $\sigma \in [u_1, u_2]$, we have:*

$$\begin{aligned}
&\frac{1-q}{1+q}\left\{\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\zeta(\lambda)d_q\lambda+\int_{\frac{u_1+u_2}{2}}^{u_2}(u_2-\lambda)^2\zeta(\lambda)d_q\lambda\right\} \qquad (31)\\
&\le \int_{u_1}^{u_2}\left(\Psi(q^2\lambda)+(q-1)\Psi(q\lambda)-\Psi(\lambda)\right)d_q\lambda+(1-q)\left(u_2\Psi(u_2)-u_1\Psi(u_1)\right)\\
&+\frac{2(1-q)}{1+q}\left(u_2\Psi(qu_2)-u_1\Psi(qu_1)-(u_2-u_1)\Psi\left(\frac{q(u_1+u_2)}{2}\right)\right)\\
&\le \frac{1-q}{1+q}\left\{\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\eta(\lambda)d_q\lambda+\int_{\frac{u_1+u_2}{2}}^{u_2}(u_2-\lambda)^2\eta(\lambda)d_q\lambda\right\}.
\end{aligned}$$

Proof. Under given condition on $D_q^2\Psi$, for $\lambda \in [u_1, \frac{u_1+u_2}{2}]$ the following q-integral inequality holds:

$$\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2D_q^2\Psi(\lambda)d_q\lambda-\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\zeta(\lambda)d_q\lambda \ge 0. \qquad (32)$$

We use q-integration by parts by applying (3), to get the following inequality:

$$\begin{aligned}
&\left(\frac{u_2-u_1}{2}\right)^2D_q\Psi\left(\frac{u_1+u_2}{2}\right)-\int_{u_1}^{\frac{u_1+u_2}{2}}D_q\Psi(q\lambda)((1+q)\lambda-2u_1)d_q\lambda \qquad (33)\\
&-\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\zeta(\lambda)d_q\lambda \ge 0.
\end{aligned}$$

On simplifying, the above inequality takes the following form

$$\begin{aligned}
&\left(\frac{u_2-u_1}{2}\right)^2D_q\Psi\left(\frac{u_1+u_2}{2}\right)-(1+q)\int_{u_1}^{\frac{u_1+u_2}{2}}\lambda D_q\Psi(q\lambda)d_q\lambda \qquad (34)\\
&+2u_1\int_{u_1}^{\frac{u_1+u_2}{2}}D_q\Psi(q\lambda)d_q\lambda-\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\zeta(\lambda)d_q\lambda \ge 0.
\end{aligned}$$

Now, using the formula (2) we get

$$\begin{aligned}
&\left(\frac{u_2-u_1}{2}\right)^2D_q\Psi\left(\frac{u_1+u_2}{2}\right)-\frac{q+1}{q-1}\int_{u_1}^{\frac{u_1+u_2}{2}}\left(\Psi(q^2\lambda)-\Psi(\lambda)\right)d_q\lambda \qquad (35)\\
&+(q+1)\int_{u_1}^{\frac{u_1+u_2}{2}}\lambda D_q\Psi(\lambda)d_q\lambda+2u_1\left(\Psi\left(\frac{q(u_1+u_2)}{2}\right)-\Psi(qu_1)\right)\\
&-\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\zeta(\lambda)d_q\lambda \ge 0.
\end{aligned}$$

Applying formula (3) for doing q-integration by parts of the first term in second line of the above inequality we have

$$\begin{aligned}&\frac{(u_2-u_1)^2}{4}D_q\Psi\left(\frac{u_1+u_2}{2}\right)-\frac{q+1}{q-1}\int_{u_1}^{\frac{u_1+u_2}{2}}\left(\Psi(q^2\lambda)+(q-1)\Psi(q\lambda)-\Psi(\lambda)\right)d_q\lambda \\ &+(q+1)\left(\frac{u_1+u_2}{2}\Psi\left(\frac{u_1+u_2}{2}\right)-u_1\Psi(u_1)\right)+2u_1\left(\Psi\left(\frac{q(u_1+u_2)}{2}\right)-\Psi(qu_1)\right) \\ &-\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\zeta(\lambda)d_q\lambda\geq 0.\end{aligned} \tag{36}$$

Furthermore, for $\lambda\in[\frac{u_1+u_2}{2},u_2]$ we also have

$$\int_{\frac{u_1+u_2}{2}}^{u_2}(u_2-\lambda)^2D_q^2\Psi(\lambda)d_q\lambda-\int_{\frac{u_1+u_2}{2}}^{u_2}(u_2-\lambda)^2\zeta(\lambda)d_q\lambda\geq 0. \tag{37}$$

Applying q-integration by parts twice via formula (3) and using (2), the forthcoming inequality is formulated

$$\begin{aligned}&-\frac{(u_2-u_1)^2}{4}D_q\Psi\left(\frac{u_1+u_2}{2}\right)-\frac{q+1}{q-1}\int_{\frac{u_1+u_2}{2}}^{u_2}\left(\Psi(q^2\lambda)+(q-1)\Psi(q\lambda)-\Psi(\lambda)\right)d_q\lambda \\ &+(q+1)\left(u_2\Psi(u_2)-\frac{u_1+u_2}{2}\Psi\left(\frac{u_1+u_2}{2}\right)\right)+2u_2\left(\Psi(qu_2)-\Psi\left(\frac{q(u_1+u_2)}{2}\right)\right) \\ &-\int_{\frac{u_1+u_2}{2}}^{u_2}(u_2-\lambda)^2\zeta(\lambda)d_q\lambda\geq 0.\end{aligned} \tag{38}$$

Inequalities (36) and (38) are added, to obtain the next inequality.

$$\begin{aligned}&\int_{u_1}^{u_2}\left(\Psi(q^2\lambda)+(q-1)\Psi(q\lambda)-\Psi(\lambda)\right)d_q\lambda+(1-q)\left(u_2\Psi(u_2)-u_1\Psi(u_1)\right) \\ &+\frac{2(1-q)}{1+q}\left(u_2\Psi(qu_2)-u_1\Psi(qu_1)-(u_2-u_1)\Psi\left(\frac{q(u_1+u_2)}{2}\right)\right) \\ &\geq\frac{1-q}{1+q}\left(\int_{\frac{u_1+u_2}{2}}^{u_2}(u_2-\lambda)^2\zeta(\lambda)d_q\lambda+\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2\zeta(\lambda)d_q\lambda\right).\end{aligned} \tag{39}$$

The proof of second inequality is on the same way; here we use the non-negativity of terms $(\lambda-u_1)^2(\eta(\lambda)-D_q^2\Psi(\lambda))$ and $(u_2-\lambda)^2(\eta(\lambda)-D_q^2\Psi(\lambda))$, doing q-integration we get:

$$\int_{u_1}^{\frac{u_1+u_2}{2}}(\lambda-u_1)^2(\eta(\lambda)-D_q^2\Psi(\lambda))d_q\lambda+\int_{\frac{u_1+u_2}{2}}^{u_2}(u_2-\lambda)^2(\eta(\lambda)-D_q^2\Psi(\lambda))d_q\lambda\geq 0. \tag{40}$$

By using q-integration via formulas (3) and (2), the second inequality of (31) can be computed. □

Theorem 2.4. *The following q-integral inequality holds under suppositions of Theorem 2.3:*

$$\begin{aligned}&\frac{(1-q)(u_2-u_1)^2}{1+q}\left(\frac{1}{(u_2-u_1)^2}\int_{u_1}^{u_2}\left(\lambda-\frac{u_1+u_2}{2}\right)^2\zeta(\lambda)d_q\lambda\right.\\&\left.-\frac{D_q\Psi(u_2)-D_q\Psi(u_1)}{4}-\frac{(q+1)(u_2\Psi(u_2)-u_1\Psi(u_1))}{(u_2-u_1)^2}\right)\\&\leq\int_{u_1}^{u_2}\left(\Psi(q^2\lambda)+(q-1)\Psi(q\lambda)-\Psi(\lambda)\right)d_q\lambda\leq\frac{(1-q)(u_2-u_1)^2}{1+q}\\&\times\left(\int_{u_1}^{u_2}\eta(\lambda)d_q\lambda-\frac{D_q\Psi(u_2)-D_q\Psi(u_1)}{4}-\frac{(q+1)(u_2\Psi(u_2)-u_1\Psi(u_1))}{(u_2-u_1)^2}\right).\end{aligned}\tag{41}$$

Proof. Let $\lambda\in[u_1,u_2]$ and by using given condition on $D_q^2\Psi$, we have

$$\int_{u_1}^{u_2}\left(\lambda-\frac{u_1+u_2}{2}\right)^2\eta(\lambda)d_q\lambda-\int_{u_1}^{u_2}\left(\lambda-\frac{u_1+u_2}{2}\right)^2D_q^2\Psi(\lambda)d_q\lambda\geq 0.\tag{42}$$

Doing q-integration by parts and for $\lambda\in[u_1,u_2]$ using $\left(\lambda-\frac{u_1+u_2}{2}\right)^2\leq\left(u_2-u_1\right)^2$, after some rearrangement of terms the following inequality is obtained

$$\begin{aligned}&\frac{(u_2-u_1)^2(D_q\Psi(u_2)-D_q\Psi(u_1))}{4}-(q+1)\int_{u_1}^{u_2}\lambda D_q\Psi(q\lambda)d_q\lambda\\&+(u_2-u_1)\int_{u_1}^{u_2}\lambda D_q\Psi(q\lambda)d_q\lambda\leq(u_2-u_1)^2\int_{u_1}^{u_2}\eta(\lambda)d_q\lambda.\end{aligned}\tag{43}$$

Using formula (2) and q-integration we will get the following inequality:

$$\begin{aligned}&\int_{u_1}^{u_2}\left(\Psi(q^2\lambda)+(q-1)\Psi(q\lambda)-\Psi(\lambda)\right)d_q\lambda\leq\frac{(1-q)(u_2-u_1)^2}{1+q}\\&\times\left(\int_{u_1}^{u_2}\eta(\lambda)d_q\lambda-\frac{D_q\Psi(u_2)-D_q\Psi(u_1)}{4}-\frac{(q+1)(u_2\Psi(u_2)-u_1\Psi(u_1))}{(u_2-u_1)^2}\right).\end{aligned}\tag{44}$$

By using non-negativity of $\left(\lambda-\frac{u_1+u_2}{2}\right)^2(D_q^2\Psi(\lambda)-\zeta(\lambda))$ and taking q-integration we will get

$$\int_{u_1}^{u_2}\left(\lambda-\frac{u_1+u_2}{2}\right)^2D_q^2\Psi(\lambda)d_q\lambda\geq\int_{u_1}^{u_2}\left(\lambda-\frac{u_1+u_2}{2}\right)^2\zeta(\lambda)d_q\lambda.\tag{45}$$

Doing q-integration by parts we get

$$\begin{aligned}&\frac{(u_2-u_1)^2(D_q\Psi(u_2)-D_q\Psi(u_1))}{4}-(q+1)\int_{u_1}^{u_2}\lambda D_q\Psi(q\lambda)d_q\lambda\\&+(u_2-u_1)\int_{u_1}^{u_2}\lambda D_q\Psi(q\lambda)d_q\lambda\geq\int_{u_1}^{u_2}\left(\lambda-\frac{u_1+u_2}{2}\right)^2\zeta(\lambda)d_q\lambda.\end{aligned}\tag{46}$$

Using formula (2) and q-integration we will get the first inequality of (41). □

3. **Conclusion**

This research established Ostrowski and Ostrowski-Grüss type inequality in quantum calculus. Those were proved by applying boundedness of first order q-derivative. By analyzing second order q-derivatives to be bounded, two q-integral inequalities similar to error estimations of the Hadamard inequality were derived.

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