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ON THE POSITIVITY OF $2D$ FRACTIONAL LINEAR ROESSER MODEL USING THE CONFORMABLE DERIVATIVE

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In this paper, we focus on the positivity of 2D fractional linear hybrid systems (continuous- discrete time). Precisely, we investigate a new class of systems that are described by the Roesser model, in which the considered derivative is the conformable fractional derivative. A general solution is then given for these classes as well as necessary and sufficient conditions for their positivity. Finally, some numerical examples are given to illustrate our results.

Keywords: Positivity, Two-dimensional systems, Roesser model, Fractional calculus, Conformable derivative.

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1. Introduction

In many physical systems, the variables are by nature positive, but the usual models, in particular linear ones, do not generally integrate this constraint. Specific models have been developed by many scientists, including compartmental models for medicine and biology, electrical models (RLC circuits), other models appear in the fields of social sciences, micro- and micro-economics, manufacturing, communication, information science, and industrial processes involving chemical reactor. [9, 11, 14, 18, 19, 21]. In control theory, the two dimensional systems are a noteworthy type of physical system that propagate the state in two independent directions. The models introduced by Roesser are the most widely used models for linear systems in two dimensions. Several authors introduced an overview of the positive two dimensional system in control theory in the literature [9–11, 14, 15].

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Due to its usefulness in describing the derivation and integration of fractional orders in the case of two dimensional systems, fractional calculus is currently very popular among researchers in basic mathematical sciences and control theory engineers. Moreover, the 2D positive fractional systems in control theory has been introduced and developed in [11, 14, 16, 17]. The positivity conditions as well as, the asymptotic stability test and the stabilization, were investigated in the standard form and in the fractional forms; see e.g.; [1, 3–5, 7, 8, 13]. To overcome certain problems in fractional calculus, Khalil *et al.* [22] presented a new fractional derivative called the conformable derivative. In [2], Abdeljawad gave some fundamental properties if the conformable fractional calculus. Kaczorek in [21] investigated the positivity and the stability of a 1D fractional linear continuous-time systems defined via the conformable derivative. Meanwhile, Thabet et al. in [25] have proposed a new work concerning the resolution of a non linear system using the conformable fractional derivative. Some other work that illustrate the importance and applicability of the new conformable fractional derivative are presented in [6, 23].

This paper introduce a new class of 2D fractional linear continuous discrete-time systems (commonly called hybrid systems) that is described by the Roesser model using the conformable derivative. The solutions are calculated and the positivity conditions are derived.

In order to analyze the conformable Roesser models, this work aims to propose an efficient analytical method. The remainder of this paper is organized as follows. In Section 2, we introduce some preliminaries concerning the conformable fractional calculus which are discussed in this article. Section 3 discusses an effective approach for solving the positive fractional continuous discrete-time linear system described by the Roesser model and the conformable fractional derivative. In Section 4 the main result concerning the positivity conditions will be derived. The usefulness of the proposed approach is demonstrated with some simulation results using numerical examples.

Notations

The following notations will be used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+^{n\times 1} = \mathbb{R}_+^n$ the space vectors of n non-negatives real entries, $\mathbb{R}_+^{n\times m}$ the space of the matrices with non-negatives real entries and $\mathbb{N}$ is the set of natural number.

2. **Preliminaries**

In this section, we present the definition of the Metzler matrices and the new fractional derivative of order α where $\alpha \in [0, 1[$ called the conformable derivative, see [22] and [2] for more details.

Definition 2.1. *$A = (a_{ij})_{i,j}$ is called a Metzler matrix if all its off-diagonal entries are positive i.e.: $a_{i,j} \geq 0$ for all $i \neq j$.*

We denote by $\mathbf{M}_n$ the set of the real Metzler matrices with dimension $n \geq 1$.

We recall the definition of the conformable derivative.

Definition 2.2. *Let f be a function $f : [0, \infty[\longrightarrow \mathbb{R}$, the fractional conformable derivative of the function f is defined by the following relation:*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0. \tag{1}$$

The following theorem reveals some properties based on α- differentiable functions.

Theorem 2.1. *[22]*

Let f and g be defined on $[0, \infty[$ are α-differentiable, $0 < \alpha < 1$. Then, for all $a, b \in \mathbb{R}$ we have the following relations:

$$\frac{d^\alpha}{dt^\alpha}\big[af(t) + bg(t)\big] = a\frac{d^\alpha f(t)}{dt^\alpha} + b\frac{d^\alpha g(t)}{dt^\alpha},$$

$$\frac{d^\alpha}{dt^\alpha}\big[f(t)g(t)\big] = f(t)\frac{d^\alpha g(t)}{dt^\alpha} + g(t)\frac{d^\alpha f(t)}{dt^\alpha},$$

$$\frac{d^\alpha}{dt^\alpha}\left[\frac{f(t)}{g(t)}\right] = \frac{g(t)\frac{d^\alpha f(t)}{dt^\alpha} - f(t)\frac{d^\alpha g(t)}{dt^\alpha}}{\big[g(t)\big]^2},$$

$$\frac{d^\alpha f(t)}{dt^\alpha} = t^{1-\alpha}\frac{df(t)}{dt},$$

$$\frac{d^\alpha t^q}{dt^\alpha} = qt^{q-\alpha} \quad , \forall q \in \mathbb{R},$$

$$\frac{d^\alpha e^{qt}}{dt^\alpha} = qt^{1-\alpha}e^{qt} \quad , \forall q \in \mathbb{R}.$$

3. The 2D conformable fractional hybrid Roesser ant its solution

Consider a class of 2D fractional continuous discrete-time linear Roesser model that propagates the state in two independent directions (horizontal and vertical axes) and it is described by the state space equations,

$$\frac{d^\alpha x^h(t,i)}{dt^\alpha} = A_{11}x^h(t,i) + A_{12}x^v(t,i) + B_1u(t,i), \tag{2}$$

$$x^v(t,i+1) = A_{21}x^h(t,i) + A_{22}x^v(t,i) + B_2u(t,i), \tag{3}$$

with $t > 0$, $i \in \mathbb{N}$ and $0 < \alpha \leq 1$

Here, $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}^{n_2 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$ are the state matrices, $x^h(t,i)$ and $x^v(t,i)$ are the horizontal and vertical state variables respectively, $B_1 \in \mathbb{R}^{n_1 \times m}$, $B_2 \in \mathbb{R}^{n_2 \times m}$ and $x^h(0,i) \in \mathbb{R}^{n_1}$, $\forall i \in \mathbb{Z}_+$ and $x^v(t,0) \in \mathbb{R}^{n_2}$, $\forall t \in \mathbb{R}_+$ are the initial conditions.

The fractional differential operator $\frac{d^\alpha x^h(t,i)}{dt^\alpha}$ is defined by

$$\frac{d^\alpha x^h(t,i)}{dt^\alpha} = \lim_{\varepsilon \to 0} \frac{x^h(t+\varepsilon t^{1-\alpha}, i) - x^h(t,i)}{\varepsilon}, \quad t > 0, i \in \mathbb{N}^*. \tag{4}$$

Since the equations (2) and (3) define a new class of systems, the resolution is necessary to deduce the positivity conditions. Therefore, we firstly prove the following result.

Theorem 3.1. *The solution $(x^h(t,i), x^v(t,i))$ of the equations (2) and (3) has the following form*

$$\begin{cases} x^h(t,0) = \Phi_\alpha(t)x^h(0,0) + C_{\alpha t}x^v(t,0) + Z_{\alpha t}u(t,0) & for\ i = 0 \\ x^h(t,i) = \Phi_\alpha(t)x^h(0,i) + \sum_{k=0}^{i-1} C_{\alpha t}(A_{21}C_{\alpha t} + A_{22})^{i-(k+1)} \big[A_{21}\Phi_\alpha(t)x^h(0,k) & \\ \quad + (A_{21}Z_{\alpha t} + B_2)u(t,k)\big] + C_{\alpha t}(A_{21}C_{\alpha t} + A_{22})^i x^v(t,0) + Z_{\alpha t}u(t,i) & for\ i \geq 1 \end{cases} \tag{5}$$

and

$$x^v(t,i) = \sum_{k=0}^{i-1} (A_{21}C_{\alpha t} + A_{22})^{i-(k+1)} \Big[A_{22}\Phi_\alpha(t)x^h(0,k) + (A_{21}Z_{\alpha t} + B_2)u(t,k)\Big] + (A_{21}C_{\alpha t} + A_{22})^i x^v(t,0) \quad for \quad i \geq 1, \tag{6}$$

where the operators $C_{\alpha t}$ and $Z_{\alpha t}$ are defined by the relations:

$$C_{\alpha t} = \int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha - \tau^\alpha)} A_{12} \tau^{\alpha-1} d\tau, \tag{7}$$

$$Z_{\alpha t} = \int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha - \tau^\alpha)} B_1 \tau^{\alpha-1} d\tau \tag{8}$$

and the transition matrices are defined by the following relation:

$$\Phi_\alpha(t) = e^{\frac{A_{11}}{\alpha}t^\alpha}. \tag{9}$$

Proof. By induction, the solution of the equation (2) is given by

$$x^h(t,i) = e^{\frac{A_{11}}{\alpha}t^\alpha} x^h(0,i) + \int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha - \tau^\alpha)} \Big[A_{12}x^v(\tau,i) + B_1u(\tau,i)\Big] \tau^{\alpha-1} d\tau, \tag{10}$$

and for $i = 0$

$$x^h(t,0) = e^{\frac{A_{11}}{\alpha}t^\alpha} x^h(0,0) + \int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha - \tau^\alpha)} \big[A_{12}x^v(\tau,0) + B_1u(\tau,0)\big] \tau^{\alpha-1} d\tau. \tag{11}$$

Following the equation (3) and for $i = 0$, we obtain

$$x^v(t,1) = A_{21}x^h(t,0) + A_{22}x^v(t,0) + B_2u(t,0), \tag{12}$$

replacing (11) in (12), we get

$$\begin{aligned} x^v(t,1) &= A_{21}\Big[e^{\frac{A_{11}}{\alpha}t^\alpha}x^h(0,0)+\int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha-\tau^\alpha)}\big[A_{12}x^v(\tau,0)+B_1u(\tau,0)\big]\tau^{\alpha-1}d\tau\Big] \\ &+A_{22}x^v(t,0)+B_2u(t,0) \end{aligned}$$

Hence

$$x^v(t,1)=A_{21}\Phi_\alpha(t)x^h(0,0)+\big(A_{21}C_{\alpha t}+A_{22}\big)x^v(t,0)+\big(A_{21}Z_{\alpha t}+B_2\big)u(t,0), \tag{13}$$

Similarly, a substitution of (13) into (10) and taking $i=1$, we get

$$x^h(t,1)=\Phi_\alpha(t)x^h(0,1)+C_{\alpha t}x^v(t,1)+Z_{\alpha t}u(t,1),$$

Therefore

$$\begin{aligned} x^h(t,1) = {} & C_{\alpha t}A_{12}\Phi_\alpha(t)x^h(0,0)+\Phi_\alpha(t)x^h(0,1)+C_{\alpha t}\big(A_{21}C_{\alpha t} \\ & +A_{22}\big)x^v(t,0)+C_{\alpha t}\big(A_{21}Z_{\alpha t}+B_2\big)u(t,0)+Z_{\alpha t}u(t,1). \end{aligned} \tag{14}$$

Suppose now that the solution is true for $i=k$, and prove that it is also true for $i=k+1$.
By the same manner and taking into account the relations (3), (5) and (6) for $i=k>1$, we have the following:

$$\begin{aligned} & A_{21}x^h(t,k)+A_{22}x^v(t,k)+B_2u(t,k)= \\ & A_{21}\Big\{\Phi_\alpha(t)x^h(0,k)+\sum_{j=0}^{k-1}C_{\alpha t}\big(A_{21}C_{\alpha t}+A_{22}\big)^{k-(j+1)}\big[A_{21}\Phi_\alpha(t) \\ & x^h(0,j)+(A_{21}Z_{\alpha t}+B_2)u(t,j)\big]+C_{\alpha t}\big(A_{21}C_{\alpha t}+A_{22}\big)^k x^v(t,0) \\ & +Z_{\alpha t}u(t,k)\Big\}+A_{22}\Big\{\sum_{j=0}^{k-1}\big(A_{21}C_{\alpha t}+A_{22}\big)^{k-(j+1)}\big[A_{21}\Phi_\alpha(t)x^h(0,j) \\ & +(A_{21}Z_{\alpha t}+B_2)u(t,j)+(A_{21}C_{\alpha t}+A_{22})^k x^v(t,0)\big]\Big\}+B_2u(t,k) \\ & =\sum_{j=0}^{k}(A_{21}C_{\alpha t}+A_{22})^{k-j}\big[A_{21}\Phi_\alpha(t)x^h(0,j)+(A_{21}Z_{\alpha t}+B_2)u(t,j)\big] \\ & +(A_{21}C_{\alpha t}+A_{22})^{k+1}x^v(t,0) \\ & =x^v(t,k+1). \end{aligned}$$

By the same procedure and using the (10),(5) and (6) for $i = k > 1$, we deduce the following relation:

$$\begin{aligned}
&\Phi_\alpha(t)x^h(0,k+1) + C_{\alpha t}x^v(t,k+1) + Z_{\alpha t}u(t,k+1) = \\
&\Phi_\alpha(t)x^h(0,k+1) + C_{\alpha t}\Big\{\sum_{j=0}^{k}(A_{21}C_{\alpha t}+A_{22})^{k-j} \\
&\big[A_{21}\Phi_\alpha(t)x^h(0,j) + (A_{21}Z_{\alpha t}+B_2)u(t,j)\big] + (A_{21}C_{\alpha t} \\
&+A_{22})^{k+1}x^v(t,0)\Big\} + Z_{\alpha t}u(t,k+1) \\
&= \Phi_\alpha(t)x^h(0,k+1) + \sum_{j=0}^{k}C_{\alpha t}(A_{21}C_{\alpha t}+A_{22})^{k-j} \\
&\big[A_{21}\Phi_\alpha(t)x^h(0,j) + (A_{21}Z_{\alpha t}+B_2)u(t,j)\big] \\
&+C_{\alpha t}(A_{21}C_{\alpha t}+A_{22})^{k+1}x^v(t,0) + Z_{\alpha t}u(t,k+1) \\
&= x^h(t,k+1)
\end{aligned}$$

□

4. **The positivity of the conformable Roesser models**

The following definitions introduce the concept of the positivity of the systems treated in Section 3, and we will test the positivity of these systems by extracting necessary and sufficient conditions.

Definition 4.1. *The two-dimensional linear hybrid system described by the Roesser model which is defined by the equations (2) and (3) is called positive if the state vectors $x^h(t,i)$ and $x^v(t,i)$ are positive for every positive initial conditions $x^h(t,0), x^h(0,i)$ and all positive entries $u(t,i)$ i.e.:*

$$x^h(t,i) \in \mathbb{R}_+^{n_1} \text{ and } x^v(t,i) \in \mathbb{R}_+^{n_2} \quad \text{where } t \in \mathbb{R}_+,\ i \in \mathbb{Z}_+,$$

for all

$$\begin{array}{lll} x^h(t,0) \in \mathbb{R}_+^{n_1} & , \ x^v(t,0) \in \mathbb{R}_+^{n_2}, & t \in \mathbb{R}_+, \\ x^h(0,i) \in \mathbb{R}_+^{n_1} & , \ x^v(0,i) \in \mathbb{R}_+^{n_2}, & i \geq 1,\ i \in \mathbb{Z}_+, \end{array} \tag{15}$$

and every $u(t,i) \in \mathbb{R}_+^m$.

Theorem 4.1. *The two-dimensional linear continuous discrete-time system described by the Roesser model and defined by the equations (2) and (3) is positive if and only if the following conditions are satisfied,*

- *$A_{12} \in \mathbb{R}_+^{n_1\times n_2}$, $A_{21} \in \mathbb{R}_+^{n_2\times n_1}$, $A_{22} \in \mathbb{R}_+^{n_2\times n_2}$, $B_1 \in \mathbb{R}_+^{n_1\times m}$ and $B_2 \in \mathbb{R}_+^{n_2\times m}$.*
- *A_{11} is a Metzler matrix*

Proof. It is well known that for all $t > 0$ and $0 < \alpha < 1$ the matrix $e^{\frac{A_{11}}{\alpha}t^\alpha}$ satisfies the following relation:

$e^{\frac{A_{11}}{\alpha}t^{\alpha}} \in \mathbb{R}_+^{n1\times n1}$ if and only if A_{11} is a Metzler matrix (see [21]).
We have

$$e^{\frac{A_{11}}{\alpha}t^{\alpha}} = \sum_{k=0}^{\infty} \frac{A_{11}^k t^{\alpha k}}{\alpha^k k!},$$

so

$$e^{\frac{A_{11}}{\alpha}t^{\alpha}} = I_n + \frac{A_{11}}{\alpha}t^{\alpha} + \frac{A_{11}^2}{2\alpha^2}t^{2\alpha} + \dots,$$

then

$$e^{\frac{A_{11}}{\alpha}t^{\alpha}} \in \mathbb{R}_+^{n_1\times n_1} \quad \text{if} \quad A_{11} \in \mathbf{M}_{n_1}, \quad 0<\alpha<1\,, \forall t>0.$$

On the other hand if, $\dfrac{A_{11}}{\alpha} \in \mathbf{M}_{n_1}$, then there exist a positive real β satisfying

$$\frac{A_{11}}{\alpha} + \beta I_n > 0.$$

This leads to

$$\Big(\frac{A_{11}}{\alpha} + \beta I_n\Big) - \big(\beta I_n\big) = -\big(\beta I_n\big) + \Big(\frac{A_{11}}{\alpha} + \beta I_n\Big),$$

Hence

$$\begin{aligned} e^{\frac{A_{11}}{\alpha}t^{\alpha}} &= e^{(\frac{A_{11}}{\alpha}+\beta I_n)t^{\alpha} - (\beta I_n)t^{\alpha}}, \\ &= e^{(\frac{A_{11}}{\alpha}+\beta I_n)t^{\alpha}} e^{-(\beta I_n)t^{\alpha}}, \\ &= e^{(\frac{A_{11}}{\alpha}+\beta I_n)t^{\alpha}} e^{-\beta I_n t^{\alpha}} \in \mathbb{R}_+^{n_1\times n_1}, \quad 0<\alpha<1. \end{aligned}$$

Therefore, $e^{(\frac{A_{11}}{\alpha}+\beta I_n)t^{\alpha}} \in \mathbb{R}_+^{n_1\times n_1}, \ \ \forall t \geq 0$.

Sufficient condition : Now Suppose that $A_{11} \in \mathbf{M}_{n_1}$, $A_{12} \in \mathbb{R}_+^{n_1\times n_2}$, $A_{21} \in \mathbb{R}_+^{n_2\times n_1}$, $A_{22} \in \mathbb{R}_+^{n_2\times n_2}$ $B_1 \in \mathbb{R}_+^{n_1\times m}$, $B_2 \in \mathbb{R}_+^{n_2\times m}$ and $u(t,i) \in \mathbb{R}_+^m$, $\forall t \geq 0$. Let

$$\frac{d^{\alpha}x^h(t,i)}{dt^{\alpha}} = A_{11}x^h(t,i) + F_1(t,i), \tag{16}$$

$$x^v(t,i+1) = A_{21}x^h(t,i) + F_2(t,i), \tag{17}$$

where

$$F_1(t,i) := A_{12}x^v(t,i) + B_1u(t,i), \tag{18}$$

and

$$F_2(t,i) := A_{22}x^v(t,i) + B_2u(t,i), \tag{19}$$

According to the equations (17), (19) and for $i=0$, we have:

$$\begin{aligned} x^v(t,1) &= A_{21}x^h(t,0) + F_2(t,0) \\ &= A_{21}x^h(t,0) + A_{22}x^v(t,0) + B_2u(t,0), \end{aligned} \tag{20}$$

If $x^v(t,1) \in \mathbb{R}_+^{n_2}$, By the same, from equations (16), (18) and for $i = 1$, we get

$$\begin{aligned}\frac{d^\alpha x^h(t,1)}{dt^\alpha} &= A_{11}x^h(t,1) + F_1(t,1) \\ &= A_{11}x^h(t,1) + A_{12}A_{21}x^h(t,0) + A_{12}A_{22}x^v(t,0) \\ &\quad + A_{12}B_2u(t,0) + B_1u(t,1).\end{aligned} \tag{21}$$

On the other hand, the solution of the equation (16) satisfies the following relation:

$$x^h(t,i) = e^{\frac{A_{11}}{\alpha}t^\alpha}x^h(0,i) + \int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha-\tau^\alpha)}F_1(\tau,i)\tau^{\alpha-1}d\tau, \tag{22}$$

So

$$x^h(t,1) = e^{\frac{A_{11}}{\alpha}t^\alpha}x^h(0,1) + \int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha-\tau^\alpha)}F_1(\tau,1)\tau^{\alpha-1}d\tau. \tag{23}$$

Hence $x^h(t,1) \in \mathbb{R}_+^{n_1}$.
Now, suppose that $x^h(t,k) \in \mathbb{R}_+^{n_1}$ and $x^v(t,k) \in \mathbb{R}_+^{n_2}$, for $k \geq 1$, $\forall t > 0$, and let's prove that according to the hypothesis of the theorem (4.1) we will have $x^h(t,k+1) \in \mathbb{R}_+^{n_1}$ and $x^v(t,k+1) \in \mathbb{R}_+^{n_2}$.
From the equations (17) and (19) we deduce that

$$\begin{aligned}x^v(t,k+1) &= A_{21}x^h(t,k) + F_2(t,k) \\ &= A_{21}x^h(t,k) + A_{22}x^v(t,k) + B_2u(t,k),\end{aligned} \tag{24}$$

Therefore $x^v(t,k+1) \in \mathbb{R}_+^{n_2}$.
On the other hand

$$\begin{aligned}\frac{d^\alpha x^h(t,k+1)}{dt^\alpha} &= A_{11}x^h(t,k+1) + F_1(t,k+1) \\ &= A_{11}x^h(t,k+1) + A_{12}x^v(t,k+1) + B_1u(t,k+1) \\ &= A_{11}x^h(t,k+1) + A_{12}\Big[A_{21}x^h(t,k) + A_{22}x^v(t,k) + B_2u(t,k)\Big] \\ &\quad + B_1u(t,k+1),\end{aligned}$$

By the hypothesis of the Theorem (4.1)

$$\begin{aligned}F_1(t,k+1) = A_{12}&\Big[A_{21}x^h(t,k) + A_{22}x^v(t,k) + B_2u(t,k)\Big] \\ &+ B_1u(t,k+1) \in \mathbb{R}_+^{n1}.\end{aligned}$$

Therefore

$$x^h(t,k+1) = e^{\frac{A_{11}}{\alpha}t^\alpha}x^h(0,k+1) + \int_0^t e^{\frac{A_{11}}{\alpha}(t^\alpha-\tau^\alpha)}F_1(\tau,k+1)\tau^{\alpha-1}d\tau,$$

Hence
$x^h(t,k+1) \in \mathbb{R}_+^{n_1}$.

Necessary condition:

Suppose that the system (2) and (3) is positive i.e.: $x^h(t,i) \in \mathbb{R}_+^{n_1}$, $x^v(t,i) \in \mathbb{R}_+^{n_2}$. Let:

$$u(t,0) = 0,\ x^v(t,0) = 0, \quad t \in \mathbb{R}_+.$$

According to the equations (2) and (5), for $i = 0$ we suppose that $x^h(0,0) = e_j$, where e_j represents the j^{th} column of the identity matrix I_{n_1}.
So, the trajectory of the system does not leave the positive orthant $\mathbb{R}_+^{n_1}$ only if

$$\begin{aligned} \left.\frac{d^\alpha x^h(t,0)}{dt^\alpha}\right|_{t=0} &= A_{11}x^h(0,0) \\ &= A_{11}\left[\Phi_\alpha(0)x^h(0,0) + C_{\alpha t}x^v(0,0) + Z_{\alpha t}u(0,0)\right] \\ &= A_{11}e_j \in \mathbb{R}_+^{n_1}, \end{aligned}$$

this implies that $a_{ij} \geq 0$ for $i \neq j$. Therefore, We deduce that $A_{11} \in \mathbf{M}_{n_1}$.
By the same manner, if $x^h(0,0) = x^v(0,0) = 0$ and $i = 0$, we will have

$$\left.\frac{d^\alpha x^h(t,0)}{dt^\alpha}\right|_{t=0} = B_1 u(0,0) \in \mathbb{R}_+^{n_1},$$

so, $B_1 \in \mathbb{R}_+^{n_1 \times m}$.
If $x^h(0,0) = 0$, $u(0,0) = 0$ and $i = 0$, So

$$\left.\frac{d^\alpha x^h(t,0)}{dt^\alpha}\right|_{t=0} = A_{12}x^v(0,0) \in \mathbb{R}_+^{n_2},$$

which implies that $A_{12} \in \mathbb{R}_+^{n_1 \times n_2}$.
Analogously, we prove the positivity of the matrices A_{21}, A_{22} and B_2. □

5. Examples and simulation

In this section we will show the applicability of the obtained result by some numerical examples.

Example 5.1. *The considered model described by the equation* (2) *and* (3) *with* $\alpha = 0.5$, *zero boundary conditions and system matrices*

$$A = \begin{bmatrix} -2 & -1 & -1 \\ 1 & -8 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$u(t,i) = 1 + i$$

Using the formula (5) *and* (6) *theorem 3.1 we obtain the following figure which represent the plot of the state variables*

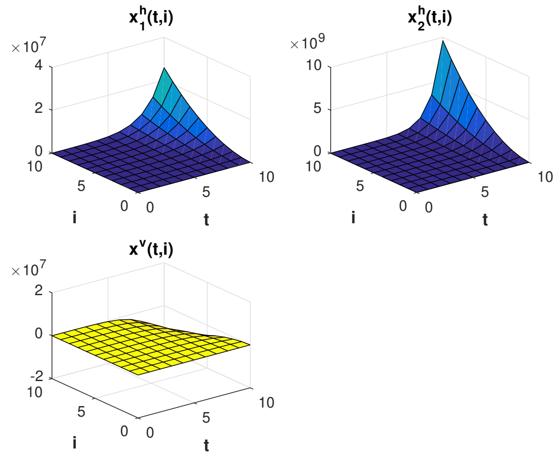

FIGURE 1. The plot of state variables

Example 5.2. *Let us consider the model described by the state space equation* (2) *and* (3) *with* $\alpha = 0.5$, *zero boundary conditions and system matrices*

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 5 & -0.6 & 0 \\ 0.2 & 1.5 & 1.8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5 \\ 0.8 \\ 5 \end{bmatrix}$$

Applying the results of theorem 3.1 and the equations (5), (6) *we obtain the plot of the state variables*

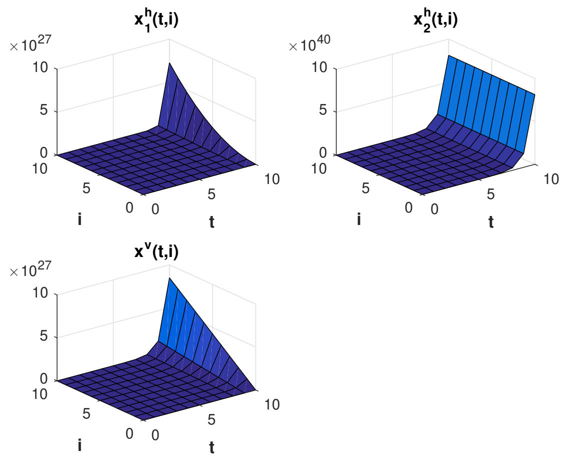

FIGURE 2. The plot of state variables

6. Concluding remarks

In this paper, a new class of two-dimensional hybrid fractional linear systems described by the Roesser model and formulated by the conformable derivative has been considered. Moreover, we established an effective computational method for solving the 2D conformable fractional Roesser models using inductive reasoning. A new necessary and sufficient positivity conditions of the considered class of models are then proposed. To reveal the accuracy and usefulness of the proposed criteria, a numerical example is tested.

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