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NEW INTEGRAL INEQUALITIES FOR RELATIVE GEOMETRICALLY SEMI-CONVEX FUNCTIONS

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In this paper, we introduce and study a new class of geometrically convex functions which is called relative geometrically semi-convex functions. We derive several new Hermite-Hadamard type inequalities for relative geometrically semi-convex functions. It is shown that this class includes the class of relative semi-convex functions as special case. Results proved in this paper may inspire future research in this field.

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1. Introduction

Convexity plays a fundamental role in different fields of pure and applied sciences. As a result the classical concepts of convex sets and convex functions have been extended and generalized in several directions, (see [1, 2, 3, 8, 9, 10, 11, 12, 13, 14, 16, 20, 21, 22]).

A significant generalization of the convex functions was the introduction of a new class which is called relative convex functions by Youness [20]. It has been shown [5, 10, 11, 12] that these relative convex functions plays an interesting role in theory of variational inequalities and optimization theory. Noor [11] proved that the minimum of a differentiable relative convex functions on the relative convex set can be characterized by a class of variational inequalities, which is known as general variational inequality. Another significant generalization of classical convex functions was the introduction of relative semi-convex function by Chen [1]. He has shown that the class of relative semi-convex functions is entirely different from the class of relative convex functions. Noor et al. [13, 15] have studied the class of relative semi-convex functions and derived many results for this class. Noor et al. [14] introduced and studied the class of relative geometrically

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convex functions. It is shown that this class contains the class of geometrically convex functions.

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

which is called Hermite-Hadamard inequality. In recent years, much attention has been given to derive the Hermite-Hadamard type inequalities for various types of convex functions, (see [4, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19, 21, 22]).

Inspired and motivated by the ongoing research in this field, we introduce and study a new class of geometrically convex functions which is called relative geometrically semi-convex functions. This is the main motivation of this paper. We discuss several special cases, which can be obtained from relative geometrically semi-convex functions. This class of nonconvex functions is quite general and unifying one. The ideas and techniques of this paper may stimulate further research in this area. We hope that interested readers may discover novel applications of the relative geometrically semi-convex functions in other fields.

2. Preliminaries

In this section, we recall some known concepts and define the class of relative geometrically (GG) semi-convex functions and relative GA semi-convex functions.

Definition 2.1 [14]. Let $\mathcal{G} \subseteq (0,\infty)$. Then $\mathcal{G}$ is said to be relative geometrically convex set, if there exists an arbitrary function $g: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(g(x))^t (g(y))^{1-t} \in \mathcal{G}, \quad \forall g(x), g(y) \in \mathcal{G}, t \in [0,1].$$

Using $AM-GM$ inequality, we have

$$(g(x))^t (g(y))^{1-t} \leq t g(x) + (1-t) g(y), \quad \forall g(x), g(y) \in \mathcal{G}, t \in [0,1].$$

Definition 2.2 [20]. A set M_g is said to be a relative convex (g-convex) set, if there exists an arbitrary function $g: \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$tg(x) + (1-t)g(y) \in M_g, \quad \forall x, y \in \mathbb{R}^n : g(x), g(y) \in M_g, t \in [0,1]. \quad (1)$$

Recently it has been shown in [5], that if M_g is a relative convex set then it is possible that it may not be a classical convex set.

Definition 2.3. A function $f: \mathcal{G} \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be relative geometrically semi-convex function (relative GG semi-convex function) if there exists an arbitrary function $g:\mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f((g(x))^t (g(y))^{1-t}) \le (f(x))^t (f(y))^{1-t}, \quad \forall g(x), g(y) \in \mathcal{G}, t \in [0,1]. \qquad 2)$$

From (2), it follows that

$$\log f((g(x))^t (g(y))^{1-t}) \le t \log f(x) + (1-t)\log f(y), \quad \forall g(x), g(y) \in \mathcal{G}, t \in [0,1].$$

Using $AM-GM$ inequality, we have

$$f((g(x))^t (g(y))^{1-t}) \le (f(x))^t (f(y))^{1-t} \le tf(x) + (1-t)f(y).$$

This implies that every relative geometrically semi-convex function (that is relative GG semi-convex function) is also relative GA semi-convex function, but the converse is not true.

Definition 2.4. A function $f:\mathcal{G} \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be relative GA semi-convex function if there exists an arbitrary function $g:\mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f((g(x))^t (g(y))^{1-t}) \le tf(x) + (1-t)f(y), \quad \forall g(x), g(y) \in \mathcal{G}, t \in [0,1]. \qquad (3)$$

We note that if $g(x) = e^x$, then Definition 2.4 reduces to

$$f(e^{tx+(1-t)y}) \le tf(x) + (1-t)f(y), \quad \forall x, y \in \mathbb{R}, t \in [0,1]. \qquad (4)$$

Definition 2.5 [1, 13]. A function f is said to be a relative semi-convex (semi g -convex) function (that is relative AA semi-convex function) on a relative convex (g -convex) set M_g, if there exists an arbitrary function $g:\mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f((1-t)g(x) + tg(y)) \le (1-t)f(x) + tf(y), \forall g(x), g(y) \in M_g, t \in [0,1]. \qquad (5)$$

Definition 2.6. A function $f:\mathcal{G} \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be relative geometrically quasi semi-convex function if there exists an arbitrary function $g:\mathbb{R}^n \to \mathbb{R}^n$ such that

$$f((g(x))^t (g(y))^{1-t}) \le \max\{f(x), f(y)\}, \quad \forall g(x), g(y) \in M_g, t \in [0,1]. \, (6)$$

From inequalities (2), (3) and (6), it follows that

$$\begin{aligned} f((g(x))^t (g(y))^{1-t}) &\le (f(x))^t (f(y))^{1-t} \\ &\le tf(x) + (1-t)f(y) \\ &\le \max\{f(x), f(y)\}. \end{aligned}$$

Essentially using the techniques of [8], one can easily show that the following statements are equivalent:

I. f is relative geometrically semi-convex function on relative geometrically convex set.

II. $f(a)^{\log(g(b))} f(g(x))^{\log(g(a))} f(b)^{\log(g(x))} \geq f(a)^{\log(g(x))} f(g(x))^{\log(g(b))} f(b)^{\log(g(a))}$.
where $g(x) = g(a)^t g(b)^{1-t}$ and $t \in [0,1]$.

We now recall the following definitions, see [7, 8].

Definition 2.7 [7, 8]. For all $a,b \in \mathbb{R}$ and $a \neq b$, we have

I. Arithmetic Mean: $A(a,b) = \dfrac{a+b}{2}$.

II. Geometric Mean: $G(a,b) = \sqrt{ab}$.

III. Extended Logarithmic Mean

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0, \\ \dfrac{b-a}{\log b - \log a}, & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0. \end{cases}$$

Now we give an auxiliary result which plays key role in proving some of our main results.

Lemma 2.1 [15]. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. If $f' \in \mathcal{L}[g(a), g(b)]$ for $g(a), g(b) \in I$ with $g(a) < g(b)$,then

$$\frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x) - f\left(\frac{g(a)+g(b)}{2}\right)$$

$$= (g(b)-g(a))\left[\int_0^1 \mu(t) f'(tg(a) + (1-t)g(b))\mathrm{d}t\right],$$

where

$$\mu(t) = \begin{cases} t, & \left[0, \dfrac{1}{2}\right), \\ t-1, & \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

3. Main Results

Now we are in a position to derive our main results.

Theorem 3.1. Let $f, w : [g(a), g(b)] \to (0, \infty)$ be relative geometrically semi-convex functions, then

$$\frac{1}{\ln g(b) - \ln g(a)} \int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)} \mathrm{d}g(x)$$

$$\leq \alpha \frac{f(a)+f(b)}{2}\left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a))\right]^{\frac{\alpha}{1-\alpha}} + \beta \frac{w(a)+w(b)}{2}\left[L_{(\frac{1}{\beta}-1)}(w(b), w(a))\right]^{\frac{\beta}{1-\beta}}.$$

Proof. Let f and w be relative geometrically semi-convex functions. Using inequality $xy \leq \alpha x^{\frac{1}{\alpha}} + \beta y^{\frac{1}{\beta}}$, $\alpha, \beta \in [0,1], \alpha + \beta = 1$, we have

$$\frac{1}{\ln g(b) - \ln g(a)} \int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)} \mathrm{d}g(x)$$

$$= \int_0^1 f((g(a))^t (g(b))^{1-t}) w((g(a))^t (g(b))^{1-t}) \mathrm{d}t$$

$$\leq \int_0^1 \left\{ \alpha (f((g(a))^t (g(b))^{1-t})^{\frac{1}{\alpha}} + \beta (w((g(a))^t (g(b))^{1-t})^{\frac{1}{\beta}} \right\} \mathrm{d}t$$

$$\leq \int_0^1 \left\{ \alpha [(f(a))^t (f(b))^{1-t}]^{\frac{1}{\alpha}} + \beta [(w(a))^t (w(b))^{1-t}]^{\frac{1}{\beta}} \right\} \mathrm{d}t$$

$$= \alpha (f(b))^{\frac{1}{\alpha}} \int_0^1 \left(\frac{f(a)}{f(b)} \right)^{\frac{t}{\alpha}} \mathrm{d}t + \beta (w(b))^{\frac{1}{\beta}} \int_0^1 \left(\frac{w(a)}{w(b)} \right)^{\frac{t}{\beta}} \mathrm{d}t$$

$$= \alpha^2 (f(b))^{\frac{1}{\alpha}} \int_0^{\frac{1}{\alpha}} \left(\frac{f(a)}{f(b)} \right)^{u} \mathrm{d}u + \beta^2 (w(b))^{\frac{1}{\beta}} \int_0^{\frac{1}{\beta}} \left(\frac{w(a)}{w(b)} \right)^{v} \mathrm{d}v$$

$$= \alpha^2 \frac{(f(b))^{\frac{1}{\alpha}} - (f(a))^{\frac{1}{\alpha}}}{\log f(b) - \log f(a)} + \beta^2 \frac{(w(b))^{\frac{1}{\beta}} - (w(a))^{\frac{1}{\beta}}}{\log w(b) - \log w(a)}$$

$$=\alpha^2\frac{(f(b))^{\frac{1}{\alpha}}-(f(a))^{\frac{1}{\alpha}}}{f(b)-f(a)}L(f(b),f(a))+\beta^2\frac{(w(b))^{\frac{1}{\alpha}}-(w(a))^{\frac{1}{\beta}}}{w(b)-w(a)}L(w(b),w(a))$$

$$=\alpha\left[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\right]^{\frac{\alpha}{1-\alpha}}L(f(b),f(a))+\beta\left[L_{(\frac{1}{\beta}-1)}(w(b),w(a))\right]^{\frac{\beta}{1-\beta}}L(w(b),w(a))$$

$$\leq\alpha\frac{f(a)+f(b)}{2}\left[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\right]^{\frac{\alpha}{1-\alpha}}+\beta\frac{w(a)+w(b)}{2}\left[L_{(\frac{1}{\beta}-1)}(w(b),w(a))\right]^{\frac{\beta}{1-\beta}}.$$

This completes the proof. □

Theorem 3.2. Let $f,w:[g(a),g(b)]\to(0,\infty)$ be increasing and relative geometrically semi-convex functions, then,

$$\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)}f(g(x))\mathrm{d}g(x)L[w(a),w(b)]$$
$$+\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)}w\left(\frac{g(a)g(b)}{g(x)}\right)\mathrm{d}g(x)L[f(a),f(b)]$$

$$\leq\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)}f(g(x))w\left(\frac{g(a)g(b)}{g(x)}\right)\mathrm{d}g(x)+L[f(a)w(a),f(b)w(b)].$$

Proof. Let f and w be relative geometrically convex functions. Then we have

$$f((g(a))^{1-t}(g(b))^{t})\leq[f(a)]^{1-t}[f(b)]^{t}$$
$$w((g(a))^{t}(g(b))^{1-t})\leq[w(a)]^{t}[w(b)]^{1-t}.$$

Now, using $\langle x_1-x_2,x_3-x_4\rangle\geq0,(x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2<x_3<x_4$, we have

$$f((g(a))^{1-t}(g(b))^{t})[w(a)]^{t}[w(b)]^{1-t}+w((g(a))^{t}(g(b))^{1-t})[f(a)]^{1-t}[f(b)]^{t}$$

$$\leq f((g(a))^{1-t}(g(b))^{t})w((g(a))^{t}(g(b))^{1-t})+[f(a)]^{1-t}[f(b)]^{t}[w(a)]^{t}[w(b)]^{1-t}.$$

Integrating above inequalities with respect to t on $[0,1]$, we have

$$\int_0^1 f((g(a))^{1-t}(g(b))^t)[w(a)]^t[w(b)]^{1-t}\,\mathrm{d}t+\int_0^1 w((g(a))^t(g(b))^{1-t})[f(a)]^{1-t}[f(b)]^t\,\mathrm{d}t$$

$$\leq\int_0^1 f((g(a))^{1-t}(g(b))^t)w((g(a))^t(g(b))^{1-t})\mathrm{d}t+\int_0^1[f(a)]^{1-t}[f(b)]^t[w(a)]^t[w(b)]^{1-t}\,\mathrm{d}t,$$

which implies

$$\int_0^1 f((g(a))^{1-t}(g(b))^t)\mathrm{d}t\int_0^1[w(a)]^t[w(b)]^{1-t}\,\mathrm{d}t+\int_0^1 w((g(a))^t(g(b))^{1-t})\mathrm{d}t\int_0^1[f(a)]^{1-t}[f(b)]^t\,\mathrm{d}t$$

$$\leq\int_0^1 f((g(a))^{1-t}(g(b))^t)w((g(a))^t(g(b))^{1-t})\mathrm{d}t+\int_0^1[f(a)]^{1-t}[f(b)]^t[w(a)]^t[w(b)]^{1-t}\,\mathrm{d}t.$$

This implies that

$$\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x)L[w(a),w(b)]$$
$$+\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} w\left(\frac{g(a)g(b)}{g(x)}\right)\mathrm{d}g(x)L[f(a),f(b)]$$

$$\leq\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} f(g(x))w\left(\frac{g(a)g(b)}{g(x)}\right)\mathrm{d}g(x)+L[f(a)w(a),f(b)w(b)].$$

This completes the proof. □

Theorem 3.3. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° (the interior of I) and $g:\mathbb{R}\to\mathbb{R}$ be arbitrary function. Let $f'\in\mathcal{L}[g(a),g(b)]$ for $g(a),g(b)\in I$ with $g(a)<g(b)$. If $|f'|$ is decreasing and relative geometrically semi-convex function, then

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x)-f\left(\frac{g(a)+g(b)}{2}\right)\right|\leq(g(b)-g(a))|f'(b)|\left[\Psi_1(w)+\Psi_2(w)\right],$$

where

$$\frac{|f'(a)|}{|f'(b)|}=w,\ \ \Psi_1(w)=\int_0^{\frac{1}{2}} tw^t\mathrm{d}t\ \text{ and }\ \Psi_2(w)=\int_{\frac{1}{2}}^1(1-t)w^t\mathrm{d}t\,.$$

Proof. Using Lemma 2.1 and the fact that $|f'|$ is decreasing and relative geometrically semi-convex function, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$=\left|(g(b)-g(a))\left[\int_0^1 \mu(t) f'(tg(a)+(1-t)g(b))\mathrm{d}t\right]\right|$$

$$\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}} |t|\,|f'((g(a))^t(g(b))^{1-t})|\,\mathrm{d}t + \int_{\frac{1}{2}}^{1} |(t-1)|\,|f'((g(a))^t(g(b))^{1-t})|\,\mathrm{d}t\right]$$
$$\leq (g(b)-g(a))|f'(b)|\left[\int_0^{\frac{1}{2}}\left\{t\left(\frac{|f'(a)|}{|f'(b)|}\right)^t\right\}\mathrm{d}t + \int_{\frac{1}{2}}^{1}\left\{(1-t)\left(\frac{|f'(a)|}{|f'(b)|}\right)^t\right\}\mathrm{d}t\right]$$
$$\leq (g(b)-g(a))|f'(b)|\left[\int_0^{\frac{1}{2}} tw^t\mathrm{d}t + \int_{\frac{1}{2}}^{1}(1-t)w^t\mathrm{d}t\right].$$

This completes the proof. □

Theorem 3.4. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g:\mathbb{R}\to\mathbb{R}$ be arbitrary function. Let $f' \in \mathcal{L}[g(a),g(b)]$ for $g(a), g(b) \in I$ with $g(a) < g(b)$. If $|f'|^q$ is decreasing and relative geometrically semi-convex function for $p,q>1, \frac{1}{p}+\frac{1}{q}=1$, then

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\leq (g(b)-g(a))|f'(b)|\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}[(\Phi_1(w))^{\frac{1}{q}} + (\Phi_2(w))^{\frac{1}{q}}],$$

where

$$\frac{|f'(a)|}{|f'(b)|} = w,\ \Phi_1(w) = \int_0^{\frac{1}{2}} w^{qt}\mathrm{d}t \ \text{ and } \ \Phi_2(w) = \int_{\frac{1}{2}}^{1} w^{qt}\mathrm{d}t\,.$$

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|f'|^q$ is decreasing and relative geometrically semi-convex function, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x)-f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$=\left|(g(b)-g(a))\left[\int_0^1 \mu(t)f'(tg(a)+(1-t)g(b))\mathrm{d}t\right]\right|$$

$$\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}}|t||f'((g(a))^t(g(b))^{1-t})|\,\mathrm{d}t+\int_{\frac{1}{2}}^1|(t-1)||f'((g(a))^t(g(b))^{1-t})|\,\mathrm{d}t\right]$$
$$\leq (g(b)-g(a))\left[\left(\int_0^{\frac{1}{2}} t^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'((g(a))^t(g(b))^{1-t})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1|t-1|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'((g(a))^t(g(b))^{1-t})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$\leq (g(b)-g(a))|f'(b)|\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}} w^{qt}\mathrm{d}t\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 w^{qt}\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

□

Theorem 3.5. Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° (the interior of I) and $g:\mathbb{R}\to\mathbb{R}$ be arbitrary function. Let $f'\in\mathcal{L}[g(a),g(b)]$ for $g(a),g(b)\in I$ with $g(a)<g(b)$. If $|f'|^q$ is decreasing and relative geometrically semi-convex function, then, for $q>1$, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x)-f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\leq (g(b)-g(a))|f'(b)|\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\Psi_1(w;q)^{\frac{1}{q}}+\Psi_2(w;q)^{\frac{1}{q}}\right],$$

where

$$\frac{|f'(a)|}{|f'(b)|}=w \quad , \quad \Psi_1(w;q)^{\frac{1}{q}}=\int_0^{\frac{1}{2}} tw^{qt}\mathrm{d}t \quad \text{and}$$

$$\Psi_2(w;q)^{\frac{1}{q}}=\int_{\frac{1}{2}}^{1}(1-t)w^{qt}\mathrm{d}t .$$

Proof. Using Lemma 2.1, power mean inequality and the fact that $|f'|^q$ is decreasing and relative geometrically semi-convex function, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))\mathrm{d}g(x)-f\left(\frac{g(a)+g(b)}{2}\right)\right|$$

$$=\left|(g(b)-g(a))\left[\int_0^1 \mu(t)f'(tg(a)+(1-t)g(b))\mathrm{d}t\right]\right|$$

$$\le (g(b)-g(a))\left[\int_0^{\frac{1}{2}}|t||f'((g(a))^t(g(b))^{1-t})|\mathrm{d}t+\int_{\frac{1}{2}}^{1}|(t-1)||f'((g(a))^t(g(b))^{1-t})|\mathrm{d}t\right]$$

$$\le (g(b)-g(a))\left[\left(\int_0^{\frac{1}{2}} t\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t||f'((g(a))^t(g(b))^{1-t})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}|t-1||f'((g(a))^t(g(b))^{1-t})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$\le (g(b)-g(a))|f'(b)|\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{1}{2}} tw^{qt}\mathrm{d}t\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(1-t)w^{qt}\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$=(g(b)-g(a))|f'(b)|\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\Psi_1(w;q)^{\frac{1}{q}}+\Psi_2(w;q)^{\frac{1}{q}}\right].$$

This completes the proof.

For $q=1$ Theorem 3.5 reduces to Theorem 3.3.

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