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FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH A WEAKLY SINGULAR KERNEL BY USING BLOCK PULSE FUNCTIONS

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In this paper, a numerical method based on Block pulse functions (BPFs) is proposed for fractional integro-differential equations with a weakly singular kernel. The BPFs expansion and its fractional operational matrix along with collocation method are utilized to reduce fractional integro-differential equations with weakly singular kernel into a system of algebraic equations. The error estimate and convergence analysis of the proposed method is investigated. In order to show the effectiveness and accuracy of the proposed method, it is applied to some benchmark problems. The numerical results are compared with other methods existing in the recent literature.

Keywords: Fractional integro-differential equations; Weakly singular integral kernel; Block pulse functions; Collocation method, Operational matrix.

MSC2010: 47G20, 45E05, 65L60.

1. Introduction

Recently, different basis functions such as piecewise constant orthogonal functions, wavelets basis, orthogonal polynomials and Sine-Cosin functions have been used to estimate the solution of integral equations [1–13]. The BPFs as a set of piecewise constant orthogonal functions, have been studied and applied as a useful tool in the synthesis, analysis and other problems of control. Because of their clearness in expressions and their simplicity in formulations, these functions may have definite advantages for problems involving integrals and derivatives [14, 15].

Fractional integro-differential equations with a weakly singular kernel are used in modelling different physical processes. For example, these kind of integro-differential equations are used in the heat conduction problem, radiative equilibrium, elasticity and fracture mechanics [13, 16–18]. Therefore, numerical solution of Integro-differential equations equations have been investigated by many authors. A collocation method based on the Bernstein polynomials is introduced for the approximate solution of a class of linear Volterra integro-differential equations with weakly singular kernel by Isik et al. [7]. Brunner [8] has used spline collocation on uniform meshes for solving solution of weakly singular Volterra integral equations. Yi and Huang [13] used the CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel. Hermite wavelet Galerkin method for solution of Fredholm integral equations with weakly singular kernel has been introduced in [20]. In [21] a Hermite-type collocation method is considered for the solution of a second-kind Volterra integral equation with a certain weakly singular kernel. The block-by-block method for solving Volterra integro-differential equations with weakly-singular kernels are investigated by Makroglou [22]. Zhao et al. [23] have been used the piecewise polynomial

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collocation methods are used for solving the fractional integro-differential equations with weakly singular kernels.

In this paper, we investigate numerical solution of the fractional integro-differential equation with a weakly singular kernel by means of BPFs and their operational matrix. Consider the fractional integro-differential equation with weakly singular kernel

$$D_*^\alpha u(t) = f(t) + \int_0^t \frac{u(s)ds}{(t-s)^\beta} + \int_0^1 k(s,t)u(s)ds, \tag{1}$$

with initial condition

$$u(0) = 0, \tag{2}$$

where $0 < \alpha, \beta < 1$, a and b are constants, $f(t)$ and $k(s,t)$ are known functions, u(t) is an unknown function and D_*^α denotes the fractional derivative defined by Caputo [25]. For deriving approximate solution of this kind of fractional integro-differential equation, we first derive an operational matrix of fractional integration for the BPFs. Then, BPFs expansion and its operational matrix along with typical collocation method are used to obtain approximate solution. The results of the proposed method are also compared with other analytical and numerical method.

2. Fractional calculus

Fractional order calculus is a branch of calculus which deal with integration and differentiation operators of non-integer order. Among the several formulations of the generalized derivative, the Riemann-Liouville and Caputo definition are most commonly used. In this section we give some necessary definitions and mathematical preliminaries of the fractional calculus which are required for establishing our results.

Definition 2.1. *A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ and a function $f_1(t) \in C[0,\infty)$ such that $f(t) = t^p f_1(t)$, and it is said to be in the space C_μ^n, $n \in \mathbb{N}$ if $f^{(n)} \in C_\mu$.*

Definition 2.2. *The Riemann-Liouville fractional integration of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$, is defined as*

$$\left(I^\alpha f\right)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases}$$

Definition 2.3. *Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as*

$$D^\alpha f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t),\ n \in \mathbb{N},\ n-1 < \alpha \leq n.$$

The Riemann-Liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, a modified fractional differential operator D_*^α was proposed by Caputo [25].

Definition 2.4. *The fractional derivative of order $\alpha > 0$ in the Caputo sense is defined as*

$$D_*^\alpha f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,\ t > 0, & 0 \leq n-1 < \alpha < n. \end{cases} \tag{3}$$

where n is an integer, $t > 0$, and $f \in C_1^n$.

Some useful relation between the Riemann-Liouvill and Caputo fractional operators is given by the following expression:

(a) $D_*^\alpha J^\alpha f(t) = f(t)$.
(b) $J^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$.
(c) $D_*^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} & \beta \geq \alpha, \\ 0 & \beta < \alpha. \end{cases}$

For more details about fractional calculus please see [25].

3. Block pulse functions (BPFs)

BPFs have been studied by many authors and also have been applied for solving different problms [4]. Here, we present a brief review of BPFs and its properties [15]. The m-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih \\ 0 & otherwise \end{cases}$$

in which $t \in [0, T)$, $i = 1, 2, ..., m$ and $h = \frac{T}{m}$. The BPFs set are disjointed with each other in the interval $[0, T)$ and

$$b_i(t)b_j(t) = \delta_{ij} b_i(t), \ i, j = 1, 2, ..., m,$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0, T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t)dt = h\delta_{ij}, \ i, j = 1, 2, ..., m.$$

As m approach to the infinity, the m-set BPFs is a complete basis for $L^2[0, T)$, i. e. so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0, T)$, can be expanded into a BPFs series as

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \left| b_i(t) \right|^2,$$

where

$$f_i = \frac{1}{h} \int_0^T b_i(t) f(t) dt.$$

Consider the first m-terms of BPFs and write them concisely as m-vector

$$B(t) = [b_1(t), \ldots, b_m(t)]^T, \ t \in [0, T),$$

then disjointness property yield

$$B(t)^T B(t) = 1, \ B(t)B(t)^T V = \tilde{V} B(t),$$

where V is an m-vector and $\tilde{V} = diag(V)$ is an $m \times m$ matrix.

3.1. Function approximation

Any absolutely integrable function $f(t)$ defined over $[0, T)$ can be expanded in BPFs as

$$f(t) = \sum_{i=1}^{\infty} f_i b_i(t), \tag{4}$$

where f_i is obtained in (**3**.1). By truncating the infinite series in (**3**.1), we get

$$f(t) \simeq f_m(t) = \sum_{i=1}^{m} f_i b_i(t) = F^T B(t), \tag{5}$$

in which

$$B(t) = [b_1(t), ..., b_m(t)]^T, \; F = [f_1, f_2, ..., f_m]^T. \tag{6}$$

Also the BPFs coefficients f_i are obtained as (**3**.1), such that the mean square error between $f(t)$ and its BPFs expansion (**3**.2) in the interval of $t \in [0,T)$ is minimal. Morever, any two dimensional function $k(s,t) \in L^2([0,T] \times [0,T])$ can be expanded with respect to BPFs such as

$$k(s,t) = B(t)^T \Pi B(t),$$

where $B(t)$ is the m-dimensional BPFs vectors respectively, and Π is the $m \times m$ BPFs coefficient matrix with (i,j)-th element

$$\Pi_{ij} = \frac{1}{h^2} \int_0^T \int_0^T k(s,t) b_i(t) b_j(s) dt ds, \; i,j = 1,2,...,m,$$

and $h = \frac{T}{m}$.

3.2. The operational matrices

Kilicman and Al Zhour [19] investigated the generalized integral operational matrix and showed that the BPFs operational matrix of fractional integration is defined as

$$J^\alpha B(t) = P^\alpha B(t) \tag{7}$$

where P^α is the $m \times m$ operational matrix of fractional integration and

$$P^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \dots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

and $\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}$. A detailed procedure for computing the operational matrix P^α can be found in [19].

3.3. Convergence analysis of BPFs expansion

Here we review the convergence and error analysis of the BPFs expansion for a continuous function.

Theorem 3.1. *Suppose that $f(t)$ satisfies a Lipschitz condition on $[0,T]$, that is*

$$\exists M > 0, \; \forall \zeta, \; \eta \in [0,T] : |f(\zeta) - f(\eta)| \le M|\zeta - \eta|.$$

Then the BPFs expansion will be convergent in the sense that $e(t)$ approaches zero as m closes to infinity. Moreover, the convergence order is one, that is

$$\|e(t)\| = \mathcal{O}(\frac{1}{m}).$$

Proof. By defining the error between $f(t)$ and its BPFs expansion over every subinterval I_i as

$$e_i(t) = f_i b_i(t) - f(t), \qquad t \in I_i \; (i = 1, \dots, m),$$

where $I_i = \left[\frac{iT}{m}, \frac{(i+1)T}{m}\right)$, we obtain

$$\|e_i(t)\|^2 = \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} (f_i b_i(t) - f(t))^2 \, dt = \frac{T}{m} (f_i - f(\eta_i))^2, \; \eta_i \in I_i, \tag{9}$$

where we used the mean value theorem for integral. From (**3**.1) and the mean value theorem, we have

$$f_i = \frac{m}{T}\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} f(t)dt = \frac{m}{T}\frac{T}{m}f(\zeta_i) = f(\zeta_i), \qquad \zeta_i \in I_i. \tag{10}$$

By substituting (**3**.7) in (**3**.6) we obtain:

$$\|e_i(t)\|^2 = \frac{T}{m}\left(f(\zeta_i) - f(\eta_i)\right)^2 \leq \frac{TM^2}{m}|\zeta_i - \eta_i|^2 \leq \frac{T^3M^2}{m^3}, \tag{11}$$

this leads to

$$\|e(t)\|^2 = \int_0^T \left(\sum_{i=1}^m e_i(t)\right)^2 dt = \int_0^T \left(\sum_{i=1}^m e_i(t)^2\right) dt + 2\sum_{i\leq j}\int_0^T e_i(t)e_j(t)dt$$

Since for $i \neq j$, $I_i \cap I_j = 0$, then

$$\|e(t)\|^2 = \int_0^T \left(\sum_{i=1}^m e_i(t)^2\right) dt = \sum_{i=1}^m \|e_i\|^2. \tag{12}$$

Substituting (**3**.8) into (**3**.9), we have:

$$\|e(t)\|^2 \leq \frac{T^3M^2}{m^2},$$

or, in other words, $\|e(t)\| = \mathcal{O}(\frac{1}{m})$. This completes the proof. □

4. **Description of the method**

In this section, we present a numerical approach based on the BPFs expansion and collocation method for solving fractional integro-differential equation with a weakly singular kernel defined in (**1**.1). To this end, we approximate functions $D_*^\alpha u(t)$ and $k(t,s)$ as

$$D_*^\alpha u(t) = C^T B(t), \;\; k(t,s) = B(t)^T K B(t), \tag{13}$$

where C is an unknown vector and K is an known matrix. By using the properties of Caputo fractional operators D_*^α and operationa matrix of fractional oreder P^α for BPFs we get

$$u(t) = J^\alpha\left(D_*^\alpha u(t)\right) = C^T J^\alpha B(t) = C^T P^\alpha B(t), \tag{14}$$

substituting Eqs. (**4**.1)-(**4**.2) into Eq. (**1**.1), we have

$$C^T B(t) = F^T B(t) + \int_0^t \frac{C^T P^\alpha B(s)ds}{(t-s)^\beta} + \int_0^t B(t)^T K B(s) C^T P^\alpha B(s)ds$$

$$= F^T B(t) + C^T P^\alpha \int_0^t \frac{B(s)ds}{(t-s)^\beta} + B(t)^T K \left(\int_0^t B(s)B(s)^T ds\right)\left(P^\alpha\right)^T C,$$

now by using Eq. (**3**), we obtain

$$C^T B(t) = F^T B(t) + C^T P^\alpha \int_0^t \frac{B(s)ds}{(t-s)^\beta} + B^T(t) K \left(P^\alpha\right)^T C. \tag{15}$$

For $(i-1)h \leq t \leq ih$, the singular integral $\int_0^t \frac{B(s)ds}{(t-s)^\beta}$ in Eq. (**4**.3) can be approximated as

$$\int_0^t \frac{B(s)ds}{(t-s)^\beta} = \left(\int_0^h \frac{ds}{(t-s)^\beta}, \int_h^{2h}\frac{ds}{(t-s)^\beta}..., \int_{(i-1)h}^t \frac{ds}{(t-s)^\beta}, 0, ..., 0\right) \tag{16}$$

$$= \left(\frac{t^{-\beta+1} - (t-h)^{-\beta+1}}{\beta - 1}, ..., \frac{(t-(i-1)h)^{-\beta+1}}{\beta-1}, 0, ..., 0\right) = V(t).$$

Substituting Eq. (**4**.4) in Eq. (**4**.3), we get

$$C^T B(t) = F^T B(t) + C^T P^{\alpha} V(t) + B^T(t) K \left(P^{\alpha}\right)^T C, \tag{17}$$

by taking the collocation points $t_i = \frac{(2i-1)h}{2}, i = 0, 1, ..., m$ and evaluating Eq. (**4**.5) we obtain the following linear system of algebraic equations for the unknown vector C

$$C^T B(t_i) - F^T B(t_i) - C^T P^{\alpha} V(t_i) - B^T(t_i) K \left(P^{\alpha}\right)^T C = 0,\ i = 0, 1, ..., m.$$

By solving this linear system and determining vector C, we can approximate solution of fractional integro-differential equation with a weakly singular kernel (**1**.1) by substituting obtained C in Eq. (**4**.2).

5. **Estimation of the error function**

Suppose $u(t)$ is the exact solution of (**1**.1) and $u_m(t)$ is the BPFs approximate solution for $u(t)$. Here, we introduce a process for estimating the error of approximate solution, i. e. $e_m(t) = u(t) - u_m(t)$. First, by using definition of fractional operator J^{α}, the fractional integro-differential equation with a weakly singular kernel (**1**.1) can be written as

$$u(s) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \int_0^{\tau} \frac{u(s) ds d\tau}{(\tau - s)^{\beta}}$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^1 \frac{k(s,\tau) u(s)}{(t-\tau)^{1-\alpha}} ds d\tau, \tag{18}$$

now consider the perturbation function $r_m(t)$ that depends only on $u_m(t)$ as

$$r_m(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \int_0^{\tau} \frac{u_m(s) ds d\tau}{(\tau - s)^{\beta}}$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \int_0^1 k(s,\tau) u_m(s) ds d\tau - u_m(t), \tag{19}$$

subtracting (**5**.2) from (**5**.1) we obtain

$$e_m(t) = r_m(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \int_0^{\tau} \frac{e_m(s) ds d\tau}{(\tau - s)^{\beta}}$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \int_0^1 k(s,\tau) e_m(s) ds d\tau,$$

this is a fractional integral equations in which the error function $e_m(t)$ is unknown. Obviously, we can apply the proposed BPFs method in previouse section for this system to find an approximation of the error function $e_m(t)$.

6. **Numerical results**

In this section, we will present numerical experiments using the BPFs collocation method described in Section **4**. The results of the proposed method are also compared with exact solution and the CAS wavelets method presened in [24]. In all examples the algorithms are performed by Maple 17 with 20 digits precision.

Example 6.1. Consider the following fractional order integro-differential equation with weakly singular kernel [24]

$$D_*^{0.25} u(t) = f(t) + \frac{1}{2} \int_0^t \frac{u(s) ds}{(t-s)^{\frac{1}{2}}} + \frac{1}{3} \int_0^1 (t-s) u(s),$$

with the initial condition $u(0) = 0$ and

$$f(t) = \frac{\Gamma(3)}{\Gamma(2.75)} t^{1.75} + \frac{\Gamma(4)}{\Gamma(3.75)} t^{2.75} - \frac{\sqrt{\pi}\Gamma(3)}{2\Gamma(3.5)} t^{2.5} - \frac{\sqrt{\pi}\Gamma(4)}{2\Gamma(4.5)} t^{3.5} - \frac{7t}{36} + \frac{3}{20}.$$

The exact solution of this equation is $u(t) = t^2 + t^3$. We have solved this fractional order integro-differential equation with weakly singular kernel by using the proposed BPFs collocation method in section **4**. The maximum absolute error and comparison between the numerical results given by the BPFs collocation method and exact solution are shown in Fig. 1. Table 2 shows the maximum absolute error for the BPFs collocation method and CAS wavelets method presened in [24]. As numerical results show, the BPFs collocation method is efficient for solving this fractional order integro-differential equation with weakly singular kernel and by increasing the number of BPFs basis, i.e. m, the maximum absolute errors decrese.

TABLE 1. Absolute error for BPFs and CAS wavelet method [24].

	BPFs			CASW		
	$m = 6$	$m = 12$	$m = 24$	$M = 1, k = 1$	$M = 1, k = 2$	$M = 1, k = 3$
$\|e\|_\infty$	1.53×10^{-1}	7.25×10^{-2}	4.67×10^{-2}	1.27×10^{-1}	6.53×10^{-2}	9.73×10^{-2}

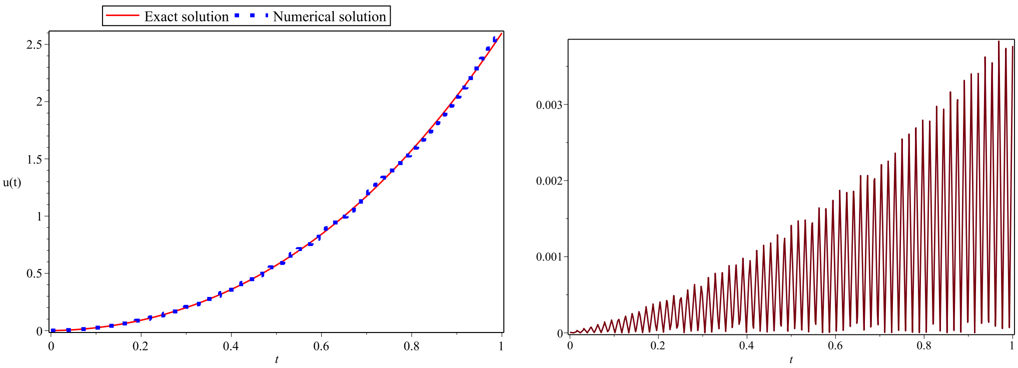

FIGURE 1. Numerical and exact solution (left) and absolute error (right) for $m = 64$.

Example 6.2. Consider the below fractional order integro-differential equation with weakly singular kernel [24]

$$D_*^{0.15} u(t) = f(t) + \frac{1}{4}\int_0^t \frac{u(s)ds}{(t-s)^{\frac{1}{2}}} + \frac{1}{7}\int_0^1 e^{t+s} u(s),$$

in which $u(0) = 0$ and

$$f(t) = \frac{\Gamma(3)}{\Gamma(2.85)} t^{1.85} - \frac{\Gamma(2)}{\Gamma(1.85)} t^{0.85} - \frac{\sqrt{\pi}\Gamma(3)}{4\Gamma(3.5)} t^{2.5} + \frac{\sqrt{\pi}\Gamma(2)}{4\Gamma(2.5)} t^{1.5} - \frac{e^{t+1} - 3e^t}{7}.$$

In this problem the exact solution of this equation is $u(t) = t^2 - t$. The proposed BPFs collocation method in section **4** are used for solving this fractional order integro-differential equation with weakly singular kernel. The maximum absolute error and comparison between the numerical results given by the BPFs collocation method and exact solution are shown in Fig. 2. A comparision between the absolute errors obtained by the BPFs collocation method and CAS wavelets method [24] is shown in Table 3. According to numerical results, it is possible to find that the BPFs collocation method is efficient in solving this fractional order integro-differential equation with weakly singular kernel.

TABLE 2. Absolute error for BPFs and CAS wavelet method [24].

	BPFs			CASW		
	$m=6$	$m=12$	$m=24$	$M=1,k=1$	$M=1,k=2$	$M=1,k=3$
$\|e\|_\infty$	1.53×10^{-1}	7.25×10^{-2}	4.67×10^{-2}	1.27×10^{-1}	6.53×10^{-2}	9.73×10^{-2}

TABLE 3. Absolute error for BPFs and CAS wavelet method [24].

	BPFs			CASW		
	$m=6$	$m=12$	$m=24$	$M=1,k=1$	$M=1,k=2$	$M=1,k=3$
$\|e\|_\infty$	3.45×10^{-1}	1.62×10^{-2}	8.61×10^{-2}	9.54×10^{-2}	7.63×10^{-2}	9.77×10^{-2}

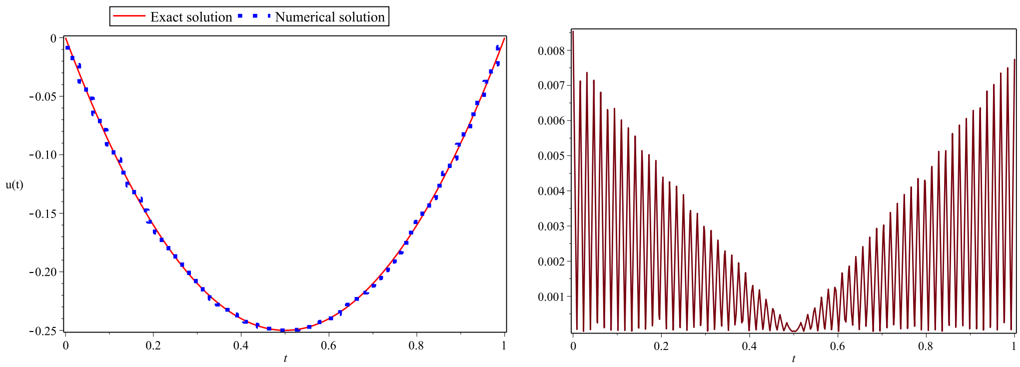

FIGURE 2. Numerical and exact solution (left) and absolute error (right) for $m=64$.

Example 6.3. Let us consider the fractional order integro-differential equation with a weakly singular kernel [24]

$$D_*^\alpha u(t) = f(t) + \int_0^t \frac{u(s)ds}{(t-s)^{\frac{1}{2}}} + \int_0^1 (t+sin(s))u(s),$$

where $u(0)=0$ and

$$f(t) = 2t - \frac{\sqrt{\pi}\Gamma(3)}{2\Gamma(3.5)}t^{2.5} - \frac{t}{3} - \cos(1) - 2\sin(1) + 2.$$

For $\alpha = 1$ the exact solution of this fractional integro-differential equation is $u(t) = t^2$. The proposed BPFs collocation method are used for solving this fractional order integro-differential equation for various values of α. Fig. 3 shows the maximum absolute error and comparison between the numerical results given by the BPFs collocation method and exact solution for $\alpha = 1$ and $m = 64$, which are in agreement with the exact solution. Moreover, The BPFs solutions for various values of α are shown in Fig. 4. From Fig. 4 we conclude that the numerical solutions derived by proposed BPFs collocation method converge to the exact solution $u(t) = t^2$ as α is close to 1.

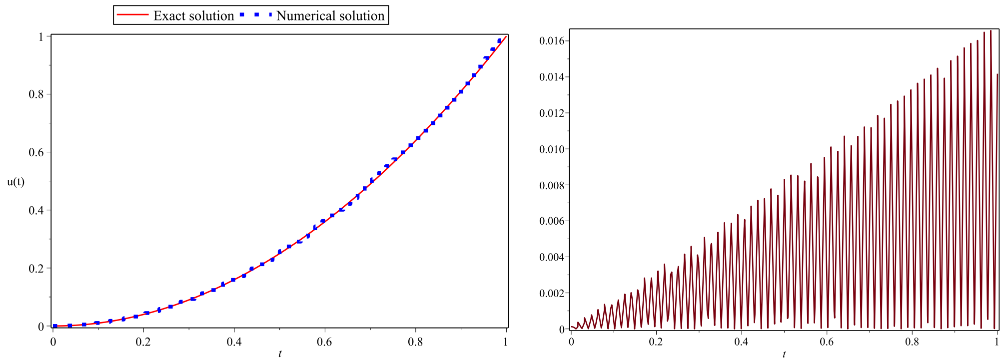

FIGURE 3. Numerical and exact solution (left) and absolute error (right) for $m = 64$.

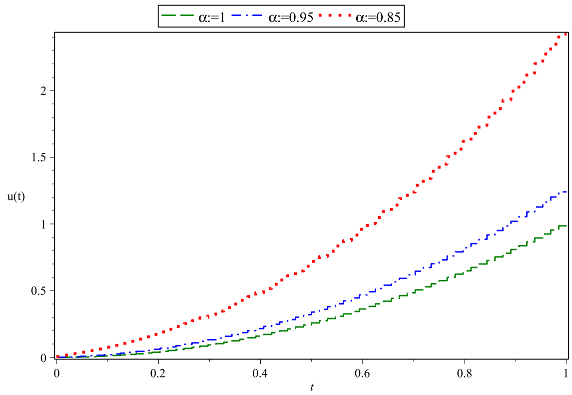

FIGURE 4. The BPFs solutions for various values of α and $m = 64$.

7. Concluding remarks

The BPFs expansion together with their operational matrix of fractional integration are used to approximate solution of this type of fractional integro-differential equation. The proposed technique have easy implementation and give satisfactory and reliable results. To reveal the superiority and efficiency of the presented method, numerical results are compared with other existing methods reported recently in the literature.

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