ROCKAFELLAR'S PROXIMAL POINT ALGORITHM FOR A FINITE FAMILY OF MONOTONE OPERATORS

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In this paper, we consider Rockafellar's proximal point algorithm with viscosity method for a finite family of monotone operators. We obtain the strong convergence of the proposed algorithm to a common zero point for a finite family of monotone operators in Hilbert spaces. The results obtained in this paper extend and improve some recent known results.

Keywords: : monotone operators, proximal point algorithm, viscosity method, zero point.

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1. Introduction

Let H be a Hilbert space and let T be a set-valued mapping with domain $D(T) = \{x \in H : Tx \neq \emptyset\}$ and range $R(T) = \{y \in H : \quad \exists x \in D(T), s.t. \quad y \in Tx\}$. Then the mapping T is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0, \qquad \forall x_i \in D(T), \forall y_i \in T(x_i),\ i = 1, 2.$$

A monotone operator T is said to be maximal monotone if the graph $G(T)$ of T,

$$G(T) = \{(x, u) \in H \times H : u \in T(x)\},$$

is not properly contained in the graph of any other monotone mapping. It is known that T is maximal iff $R(I+rT) = H$ for every $r > 0$, where $R(I+rT) = \bigcup\{z+rTz : z \in H, Tz \neq \emptyset\}$. Monotone operators have proven to be a key class of objects in modern Optimization and Analysis; see, e.g., the books [1-7] and the references therein. Let us consider the zero point problem for a monotone operator T on a real Hilbert space H, that is, finding a point $z \in H$, such that $0 \in Tz$. This problem is closely related to many kinds of important problems, such as minimization problems, saddle point problems, equilibrium problems and others. In order to approximate the solution to this problem, various types of iterative schemes have been proposed. One of the most important methods is Rockafellar proximal point algorithm [8], which generates a sequence $\{x_n\}$ according to the relation:

$$x_{n+1} = J^T_{r_n}(x_n + e_n), \tag{1.1}$$

where $J^T_r = (I + rT)^{-1}$ for all $r > 0$ is the resolvent of T and $\{e_n\}$ is a sequence of errors. Rockafellar's proved the weak convergence of the algorithm (1.1). Guler's example however shows that in an infinite dimensional Hilbert space, Rochafellar's

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algorithm has only weak convergence. To obtain the strong convergence, several authors proposed modifications of Rochafellar's proximal point algorithm (see for instance [9-18]).

In 2002, Xu [14] investigated a modified version of the initial proximal point algorithm studied by Rockafellar as follows:

$$x_{n+1} = t_n x_0 + (1 - t_n) J^T_{r_n} x_n + e_n, \tag{1.2}$$

where x_0 is the starting point of proximal point algorithm and $\{e_n\}$ is the error sequence. For $\{e_n\}$ summable, it was proved that $\{x_n\}$ is strongly convergent if $r_n \longrightarrow \infty$ and $\{t_n\} \subset (0,1)$ with $\lim_{n\longrightarrow\infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = \infty$. Algorithm (1.2) was further studied by Boikanyo and Morosanu [17] (see also [18]). Very recently, Tian and Song [19] generalized the result of Xu [14]. In fact, they show that strong convergence of (1.2) is preserved under the assumption that $\liminf_{n\longrightarrow\infty} r_n > 0$. On the other hand, Moudafi [20] introduced the viscosity approximation method for finding fixed point of a nonexpansive mapping (see [21] for further developments in both Hilbert and Banach spaces). In this paper we prove strong convergence of Rockafellar's proximal point algorithm to a common zero point for a finite family of monotone operators via viscosity method. Our result generalize some result of Tian and Song [19], Boikanyo and Morosanu [17] and many others.

2. **Preliminaries**

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converge weakly to x, and $x_n \longrightarrow x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$. This point satisfies

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C.$$

The operator P_C is called the metric projection or the nearest point mapping of H onto C. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x) \rangle \leq 0, \qquad \forall x \in H, y \in C.$$

It is well known that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. *([20]) There holds the following inequality in a Hilbert space H:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \qquad \forall x, y \in H.$$

Lemma 2.2. *Let H be a real Hilbert space. Then for $x_i \in H, a_i \in [0,1], i = 1, 2, \cdots, k$ with $\sum_{i=1}^{k} a_i = 1$, we have*

$$\|a_1 x_1 + a_2 x_2 + \cdots + a_k x_k\|^2 \leq a_1\|x_1\|^2 + a_2\|x_2\|^2 + \cdots + a_k\|x_k\|^2.$$

Proof. We prove this by mathematical induction. If $k = 2$, then we have

$$\begin{aligned}\|a_1x_1+a_2x_2\|^2 &= \langle a_1x_1+a_2x_2, a_1x_1+a_2x_2\rangle\\ &= a_1^2\|x_1\|^2+a_2^2\|x_2\|^2+2a_1a_2\,Re\langle x_1,x_2\rangle\\ &= a_1^2\|x_1\|^2+a_2^2\|x_2\|^2+a_1a_2(\|x_1\|^2+\|x_2\|^2-\|x_1-x_2\|^2)\\ &= a_1\|x_1\|^2+a_2\|x_2\|^2-a_1a_2\|x_1-x_2\|^2\\ &\le a_1\|x_1\|^2+a_2\|x_2\|^2.\end{aligned}$$

Hence the conclusion is holds. Suppose that the inequality holds for $k = n-1$. Let $a_n \neq 1$ be chosen in such a way that $\sum_{i=1}^{n} a_i = 1$. It follows from the induction hypotheses that

$$\begin{aligned}\|a_1x_1+a_2x_2+\cdots+a_nx_n\|^2 &= \|(1-a_n)\tfrac{a_1x_1+a_2x_2+\cdots+a_{n-1}x_{n-1}}{(1-a_n)}+a_nx_n\|^2\\ &\le (1-a_n)\|\tfrac{a_1x_1+a_2x_2+\cdots+a_{n-1}x_{n-1}}{(1-a_n)}\|^2+a_n\|x_n\|^2\\ &\le a_1\|x_1\|^2+a_2\|x_2\|^2+\cdots+a_{n-1}\|x_{n-1}\|^2+a_n\|x_n\|^2.\end{aligned}$$

□

Lemma 2.3. *([14]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \beta_n, \qquad n \ge 0,$$

where $\{\gamma_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ satisfy the conditions:

(i) $\gamma_n \subset [0,1]$, $\sum_{n=1}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\longrightarrow\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$,
(iii) $\beta_n \ge 0$ *for all* $n \ge 0$ *with* $\sum_{n=0}^{\infty}\beta_n < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.4. *([22]) Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{s(n)\} \subset \mathbb{N}$ such that $s(n) \longrightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$t_{s(n)} \le t_{s(n)+1}, \qquad t_n \le t_{s(n)+1}.$$

In fact

$$s(n) = \max\{k \le n : t_k < t_{k+1}\}.$$

We now recall some properties of monotone operators.

Remark 1: It is well known that for $\lambda > 0$,

(i) T is monotone if and only if the resolvent J_λ^T of T is single valued and firmly nonexpansive, (see [8]).
(ii) T is maximal monotone if and only if J_λ^T of T is single valued and firmly nonexpansive and its domain is all of H (see [8, 24]).
(iii)

$$0 \in T(x^\star) \iff x^\star \in Fix(J_\lambda^T),$$

where $Fix(J_\lambda^T)$ denotes the fixed point set of J_λ^T. Since the fixed point set of nonexpansive operators is closed convex, the projection onto the solution set

$Z = T^{-1}(0) = \{x \in D(T) : 0 \in Tx\}$ is well defined whenever $Z \neq \emptyset$. For more details, see [23, 24].

Lemma 2.5. [1]*(The Resolvent Identity) For $\lambda, \mu > 0$, there holds the identity:*

$$J_\lambda^T x = J_\mu^T(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda^T x), \qquad x \in H.$$

Lemma 2.6. *([19]) For each $\lambda > 0$, there holds the inequality:*

$$\|J_\lambda^T x - J_\lambda^T y\|^2 \leq \|x-y\|^2 - \|(x - J_\lambda^T x) - (y - J_\lambda^T x)\|^2, \qquad x, y \in R(I + \lambda T).$$

3. Main Result

Now, we state our main result.

Theorem 3.1. *Let $T_i, (i = 1, 2, ..., m)$ be a finite family of monotone operators of a Hilbert space H with $Z = \bigcap_{i=1}^m T_i^{-1}(\{0\}) \neq \emptyset$. Assume that K is a nonempty closed convex subset of H such that $\bigcap_{i=1}^m \overline{D(T_i)} \subset K \subset \bigcap_{i=1}^m R(I + rT_i)$ for all $r > 0$. Assume that f is a k-contraction of K into itself. Let $\{x_n\}$ be a sequence generated by $x_0 \in K$ and*

$$x_{n+1} = a_{n,0}f(x_n) + a_{n,1}J_{r_n}^{T_1}x_n + a_{n,2}J_{r_n}^{T_2}x_n + ... + a_{n,m}J_{r_n}^{T_m}x_n + e_n, \qquad n \geq 0,$$

where $\sum_{i=0}^m a_{n,i} = 1$. If $\{a_{n,i}\}, \{e_n\}$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(i) $\lim_{n\longrightarrow\infty} a_{n,0} = 0$ *and* $\sum_{n=0}^\infty a_{n,0} = \infty$,
(ii) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\{r_n\} \subset (0, \infty)$,
(iii) $e_n \in K$ *satisfies* $\sum_{n=0}^\infty \|e_n\| < \infty$,
(iv) $\{a_{n,i}\} \subset (b, 1) \subset (0, 1)$, $i = 1, ..., m$,

then the sequence $\{x_n\}$ converges strongly to $z \in Z$, where $z = P_Z f(z)$.

Proof. First we show that $\{x_n\}$ is bounded. In fact, let $z \in Z = \bigcap_{i=1}^m T_i^{-1}(\{0\})$. Noting that each resolvent $J_{r_n}^{T_i}$ is nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|a_{n,0}f(x_n) + a_{n,1}J_{r_n}^{T_1}x_n + a_{n,2}J_{r_n}^{T_2}x_n + ... + a_{n,m}J_{r_n}^{T_m}x_n + e_n - z\| \\
&\leq a_{n,0}\|f(x_n) - z\| + a_{n,1}\|J_{r_n}^{T_1}x_n - J_{r_n}^{T_1}z\| + ... + a_{n,m}\|J_{r_n}^{T_m}x_n - J_{r_n}^{T_m}z\| + \|e_n\| \\
&\leq a_{n,0}\|f(x_n) - z\| + a_{n,1}\|x_n - z\| + ... + a_{n,m}\|x_n - z\| + \|e_n\| \\
&\leq a_{n,0}\|f(x_n) - f(z)\| + a_{n,0}\|f(z) - z\| + (1 - a_{n,0})\|x_n - z\| + \|e_n\| \\
&\leq a_{n,0}k\|x_n - z\| + a_{n,0}\|f(z) - z\| + (1 - a_{n,0})\|x_n - z\| + \|e_n\| \\
&\leq (1 - (1-k))\|x_n - z\| + a_{n,0}\|f(z) - z\| + \|e_n\| \\
&\leq \max\{\|x_n - z\|, \tfrac{1}{1-k}\|f(z) - z\|\} + \|e_n\| \\
&\leq ... \\
&\leq \max\{\|x_0 - z\|, \tfrac{1}{1-k}\|f(z) - z\|\} + \textstyle\sum_{i=1}^n \|e_i\|.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded and we also obtain that $\{f(x_n)\}$ is bounded. Next we show that for $1 \leq i \leq m$ and for all $r > 0$, $\lim_{n\longrightarrow\infty} \|x_n - J_r^{T_i}x_n\| = 0$. By using Lemma 2.2 and Lemma 2.6, for some appropriate constant $L > 0$, we have that

$$\begin{aligned}\|x_{n+1}-z\|^2 =& \|a_{n,0}f(x_n)+a_{n,1}J_{r_n}^{T_1}x_n+a_{n,2}J_{r_n}^{T_2}x_n+...+a_{n,m}J_{r_n}^{T_m}x_n+e_n-z\|^2\\ \le& \|a_{n,0}f(x_n)+a_{n,1}J_{r_n}^{T_1}x_n+a_{n,2}J_{r_n}^{T_2}x_n+...+a_{n,m}J_{r_n}^{T_m}x_n-z\|^2+L\|e_n\|\\ \le& a_{n,0}\|f(x_n)-z\|^2+a_{n,1}\|J_{r_n}^{T_1}x_n-J_{r_n}^{T_1}z\|^2+...+a_{n,m}\|J_{r_n}^{T_m}x_n-J_{r_n}^{T_m}z\|^2+L\|e_n\|\\ \le& a_{n,0}\|f(x_n)-z\|^2+a_{n,1}\|x_n-z\|^2-a_{n,1}\|x_n-J_{r_n}^{T_1}x_n\|^2+...\\ &+a_{n,m}\|x_n-z\|-a_{n,m}\|x_n-J_{r_n}^{T_m}x_n\|^2+L\|e_n\|\\ \le& a_{n,0}\|f(x_n)-z\|^2+(1-a_{n,0})\|x_n-z\|^2-\textstyle\sum_{i=1}^m a_{n,i}\|x_n-J_{r_n}^{T_i}x_n\|^2+L\|e_n\|.\end{aligned}$$

Hence for $i=1,2,...,m$, we have

$$\begin{aligned}a_{n,i}\|x_n-J_{r_n}^{T_i}x_n\|^2 \quad &\le (1-a_{n,0})\|x_n-z\|^2-\|x_{n+1}-z\|^2+a_{n,0}\|f(x_n)-z\|^2+L\|e_n\|\\ &\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+a_{n,0}\|f(x_n)-z\|^2+L\|e_n\|.\end{aligned} \tag{3.1}$$

Now, we show that there exists a unique $z\in Z$ such that $z=P_Zf(z)$. Indeed, since $Z=\bigcap_{i=1}^m T_i^{-1}(\{0\})$ is closed and convex, we have the projection P_Z is well defined. Now, let $Q=P_Z$, we show that $Q(f)$ is a contraction of K into itself. In fact, since Q is nonexpansive,

$$\|Q(f)(x)-Q(f)(y)\|\le\|f(x)-f(y)\|\le k\|x-y\|.$$

Hence there exists a unique element $z\in Z$ such that $z=P_Zf(z)$.

In order to prove that $x_n\longrightarrow z$ as $n\longrightarrow\infty$, we consider two possible cases.

Case1. Suppose that $\{\|x_n-z\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n-z\|\}_{n\ge n_0}$ is either nondecreasing or non-increasing. Since $\|x_n-z\|$ is bounded we have $\|x_n-z\|$ is convergent. Since $\lim_{n\longrightarrow\infty}a_{n,0}=\lim_{n\longrightarrow\infty}\|e_n\|=0$ and $\{f(x_n)\}$ is bounded, from (3.1) we obtain that $\lim_{n\longrightarrow\infty}a_{n,i}\|x_n-J_{r_n}^{T_i}x_n\|^2=0$. By condition (iv), we have

$$b\|x_n-J_{r_n}^{T_i}x_n\|^2\le a_{n,i}\|x_n-J_{r_n}^{T_i}x_n\|^2,$$

which implies that $\lim_{n\longrightarrow\infty}\|x_n-J_{r_n}^{T_i}x_n\|=0$. Using the resolvent identity (Lemma 2.5), for each $r>0$ we have

$$\begin{aligned}\|x_n-J_r^{T_i}x_n\|\le& \|x_n-J_{r_n}^{T_i}x_n\|+\|J_{r_n}^{T_i}x_n-J_r^{T_i}x_n\|\\ \le& \|x_n-J_{r_n}^{T_i}x_n\|+\|J_r^{T_i}(\tfrac{r}{r_n}x_n+(1-\tfrac{r}{r_n})J_{r_n}^{T_i}x_n)-J_r^{T_i}x_n\|\\ \le& \|x_n-J_{r_n}^{T_i}x_n\|+\|\tfrac{r}{r_n}x_n+(1-\tfrac{r}{r_n})J_{r_n}^{T_i}x_n-x_n\|\\ \le& \|x_n-J_{r_n}^{T_i}x_n\|+|1-\tfrac{r}{r_n}|\|J_{r_n}^{T_i}x_n-x_n\|\longrightarrow 0, \qquad n\longrightarrow\infty.\end{aligned}$$

Next we show that there exists a unique $z\in Z$ such that $\lim sup_{n\longrightarrow\infty}\langle f(z)-z,x_n-z\rangle\le 0$, where $z=P_Zf(z)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\longrightarrow\infty}\langle f(z)-z,x_{n_i}-z\rangle=\limsup_{n\longrightarrow\infty}\langle f(z)-z,x_n-z\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i}\rightharpoonup w$. We show that $w \in Z$. Indeed,

$$\|x_{n_i} - J_r^{T_i} w\| \le \|x_{n_i} - J_r^{T_i} x_{n_i}\| + \|J_r^{T_i} x_{n_i} - J_r^{T_i} w\| \le \|x_{n_i} - J_r^{T_i} x_{n_i}\| + \|x_{n_i} - w\|,$$

which implies that

$$\limsup_{i \longrightarrow \infty} \|x_{n_i} - J_r^{T_i} w\| \le \limsup_{i \longrightarrow \infty} \|x_{n_i} - w\|.$$

By the Opial property of Hilbert space H we obtain $w = J_r^{T_i} w$, $i = 1, 2, ..., m$. Hence $w \in Z$. Therefore, it follows that

$$\limsup_{n \longrightarrow \infty} \langle f(z) - z, x_n - z \rangle = \lim_{i \longrightarrow \infty} \langle f(z) - z, x_{n_i} - z \rangle = \langle f(z) - z, w - z \rangle \le 0.$$

Finally, we show that $x_n \longrightarrow P_Z f(z)$. In fact, for some appropriate constant $M > 0$, using Lemma 2.1 and 2.2 we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \|a_{n,1} J_{r_n}^{T_1} x_n + a_{n,2} J_{r_n}^{T_2} x_n + ... + a_{n,m} J_{r_n}^{T_m} x_n + e_n - (1 - a_{n,0}) z\|^2 \\
&\quad + 2a_{n,0} \langle f(x_n) - z, x_{n+1} - z \rangle \\
&\le \|a_{n,1} J_{r_n}^{T_1} x_n + a_{n,2} J_{r_n}^{T_2} x_n + ... + a_{n,m} J_{r_n}^{T_m} x_n - (1 - a_{n,0}) z\|^2 + M\|e_n\| + 2a_{n,0} \langle f(x_n) - z, x_{n+1} - z \rangle \\
&\le (1 - a_{n,0})^2 \left(\frac{a_{n,1}}{1 - a_{n,0}} \|J_{r_n}^{T_1} x_n - z\|^2 + ... + \frac{a_{n,m}}{1 - a_{n,0}} \|J_{r_n}^{T_m} x_n - z\|^2 \right) \\
&\quad + M\|e_n\| + 2a_{n,0} \langle f(x_n) - z, x_{n+1} - z \rangle \\
&\le (1 - a_{n,0})(a_{n,1} \|x_n - z\|^2 + ... + a_{n,m} \|x_n - z\|^2) + M\|e_n\| + 2a_{n,0} \langle f(x_n) - z, x_{n+1} - z \rangle \\
&\le (1 - a_{n,0})^2 \|x_n - z\|^2 + M\|e_n\| + 2a_{n,0} \langle f(x_n) - f(z), x_{n+1} - z \rangle + 2a_{n,0} \langle f(z) - z, x_{n+1} - z \rangle \\
&\le (1 - a_{n,0})^2 \|x_n - z\|^2 + M\|e_n\| + 2a_{n,0} k \|x_n - z\| \|x_{n+1} - z\| + 2a_{n,0} \langle f(z) - z, x_{n+1} - z \rangle \\
&\le (1 - a_{n,0})^2 \|x_n - z\|^2 + M\|e_n\| + a_{n,0} k \{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\} + 2a_{n,0} \langle f(z) - z, x_{n+1} - z \rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{(1 - a_{n,0})^2 + a_{n,0} k}{1 - a_{n,0} k} \|x_n - z\|^2 + \frac{2a_{n,0}}{1 - a_{n,0} k} \langle f(z) - z, x_{n+1} - z \rangle + M\|e_n\| \\
&= \frac{1 - 2a_{n,0} + a_{n,0} k}{1 - a_{n,0} k} \|x_n - z\|^2 + \frac{a_{n,0}^2}{1 - a_{n,0} k} \|x_n - z\|^2 + \frac{2a_{n,0}}{1 - a_{n,0} k} \langle f(z) - z, x_{n+1} - z \rangle + M\|e_n\| \\
&\le \left(1 - \frac{2(1 - k) a_{n,0}}{1 - a_{n,0} k}\right) \|x_n - z\|^2 + \frac{2(1 - k) a_{n,0}}{1 - a_{n,0} k} \left\{ \frac{a_{n,0} N}{2(1 - k)} + \frac{1}{1 - k} \langle f(z) - z, x_{n+1} - z \rangle \right\} + M\|e_n\| \\
&\le (1 - \eta_n) \|x_n - z\|^2 + \eta_n \delta_n + \beta_n,
\end{aligned}$$

where $N = sup\{\|x_n - z\|^2 : n \ge 0\}$, $\eta_n = \frac{2(1-k)a_{n,0}}{1 - a_{n,0} k}$, $\beta_n = M\|e_n\|$ and

$$\delta_n = \frac{a_{n,0} N}{2(1 - k)} + \frac{1}{1 - k} \langle f(z) - z, x_{n+1} - z \rangle.$$

It is easy to see that $\eta_n \longrightarrow 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$ and $\limsup_{n \longrightarrow \infty} \delta_n \le 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Hence, by Lemma 2.3, the sequence $\{x_n\}$ converges strongly to $z = P_Z f(z)$.

Case2. Assume that $\{\|x_n - z\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{s(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$s(n) = \max\{k \in \mathbb{N}; k \leq n : \quad \|x_k - z\| < \|x_{k+1} - z\|\}.$$

Clearly, $s(n)$ is a nondecreasing sequence such that $s(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and for all $n \geq n_0$,

$$\|x_{s(n)} - z\| < \|x_{s(n)+1} - z\|.$$

From (3.1) we obtain that $\lim_{n\longrightarrow\infty} \|x_{s(n)} - J^{T_i}_{r_{s(n)}} x_{s(n)}\| = 0$. Following an argument similar to that in Case (1) we have

$$\|x_{s(n)+1} - z\|^2 \leq (1 - \eta_{s(n)})\|x_{s(n)} - z\|^2 + \eta_{s(n)}\delta_{s(n)}$$

where $\eta_{s(n)} \longrightarrow 0, \sum_{n=1}^{\infty} \eta_{s(n)} = \infty$ and $\limsup_{n\longrightarrow\infty} \delta_{s(n)} \leq 0$. Hence, by Lemma 2.3, we obtain $\lim_{n\longrightarrow\infty} \|x_{s(n)} - z\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_{s(n)+1} - z\| = 0$. Now, from Lemma 2.4 we have

$$0 \leq \|x_n - z\| \leq \max\{\|x_{s(n)} - z\|, \|x_n - z\|\} \leq \|x_{s(n)+1} - z\|.$$

Therefore $\{x_n\}$ converges strongly to $z = P_Z f(z)$. This complete the proof. □

Theorem 3.2. *Let $T_i, (i = 1, 2, ..., m)$ be a finite family of maximal monotone operators of a Hilbert space H with $Z = \bigcap_{i=1}^{m} T_i^{-1}(\{0\}) \neq \emptyset$. Assume that f is a k-contraction of H into itself. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and*

$$x_{n+1} = a_{n,0} f(x_n) + a_{n,1} J^{T_1}_{r_n} x_n + a_{n,2} J^{T_2}_{r_n} x_n + ... + a_{n,m} J^{T_m}_{r_n} x_n + e_n, \qquad n \geq 0,$$

where $\sum_{i=0}^{m} a_{n,i} = 1$. If $\{a_{n,i}\}, \{e_n\}$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(i) $\lim_{n\longrightarrow\infty} a_{n,0} = 0$, $\sum_{n=0}^{\infty} a_{n,0} = \infty$,
(ii) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\{r_n\} \subset (0, \infty)$,
(iii) $e_n \in H$ *satisfies* $\sum_{n=0}^{\infty} \|e_n\| < \infty$,
(iv) $\{a_{n,i}\} \subset (b, 1) \subset (0, 1)$, $i = 1, ..., m$,

then the sequence $\{x_n\}$ converges strongly to $z \in Z$, where $z = P_Z f(z)$.

Proof. Since T_i are maximal monotone, then T_i are monotone and satisfy the condition $\bigcap_{i=1}^{m} \overline{D(T_i)} \subset K \subset \bigcap_{i=1}^{m} R(I + rT_i)$ for all $r > 0$. Putting $K = H$, the desired result is holds. □

If we put $f(x) = u$ and $T_1 = T_2 = ... = T_m = T$ in Theorem 3.1, we obtain the following Corollary:

Corollary 3.1. *([19]) Let T be a monotone operator of a Hilbert space H with $Z = T^{-1}(\{0\}) \neq \emptyset$. Assume that K is a nonempty closed convex subset of H such that $\overline{D(T)} \subset K \subset R(I+rT)$ for all $r > 0$ and for a given point $u \in K$ and an initial value $x_0 \in K$, $\{x_n\}$ is defined by the approximate rule*

$$x_{n+1} = t_n u + (1 - t_n) J^T_{r_n} x_n + e_n.$$

If $\{t_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy

(i) $\lim_{n\longrightarrow\infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = \infty$,
(ii) $\liminf_{n\longrightarrow\infty} r_n > 0$,
(iii) $e_n \in H$ *satisfies* $\sum_{n=0}^{\infty} \|e_n\| < \infty$,

then the sequence $\{x_n\}$ *converges strongly to* $P_Z u$*, where* P_Z *is the metric projection from* H *onto* Z*.*

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