CHARACTERIZATION OF QUASICOMMUTING GRAPHS OF COMPLETELY SIMPLE SEMIGROUPS

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In this paper, we introduce the notion of the quasi-center of a semigroup. Based on this concept, we define the quasicommuting graph and the extended quasicommuting graph associated with a semigroup. We show that the extended commuting graph of a semigroup is always a subgraph of its extended quasicommuting graph. We further examine the structural properties of the quasicommuting graph for completely simple semigroups, represented as Rees matrix semigroups over a group with sandwich matrix P. Our results demonstrate that, for a completely simple semigroup, the extended quasicommuting graph coincides with its quasicommuting graph, a property that also holds for the corresponding commuting graph. Thus, the commuting graph of a completely simple semigroup is a subgraph of its quasicommuting graph. Consequently, the study extends the theory of commuting graphs of completely simple semigroups to the broader framework of quasicommuting graphs, thereby enriching the understanding of commuting-like relations within these algebraic structures.

Keywords: Semigroups, completely simple semigroups, commuting graphs, quasi-center, quasicommuting graphs.

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1. Introduction

The concepts of quasicommutative elements and quasicommutative semigroups were formulated initially by Mukherjee in 1972 in [21] and have since been developed and refined by several researchers, including subsequent studies and expositions such as Nagy's later work [22]. These ideas form a natural and productive extension of the classical concept of commutativity, enabling algebraic systems in which elements need not commute exactly, but under specific conditions. This approach provides a richer and more flexible algebraic setting, in which a broader range of structural behaviors can be examined. Over the past few decades, this perspective has significantly advanced semigroup theory and, more recently, the study of algebraic graphs associated with noncommutative structures.

A cornerstone of modern semigroup theory, particularly in the analysis of completely simple semigroups, is the Rees Theorem [24].This theorem offers a precise structural characterization by asserting that a semigroup S is completely simple if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ over a group G, endowed with binary operation

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$$

where I and Λ are index sets, and $P = (p_{\lambda j})$ is a sandwich matrix with entries in G. The Rees representation not only associates various classes of semigroups within a common framework but also provides a fundamental bridge by which numerous structural results

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and techniques from group theory can be transferred to semigroups. Consequently, the Rees Theorem remains an indispensable tool in the structural classification of semigroups and serves as the theoretical foundation for several modern generalizations.

In parallel with developments in algebra, the use of graphs associated with algebraic structures has become an increasingly powerful tool for visualizing and analyzing algebraic relations. As a natural extension of this approach, graph theory, particularly in the context of chemical graphs, has found numerous applications in the literature for the study of molecular structures and the analysis of various structural properties [8, 27]. Various kinds of algebraic graphs of rings, groups, and semigroups, such as Cayley graphs, power graphs, and zero-divisor graphs, have been extensively investigated in [3, 7, 10, 17, 18, 29], and have proved essential in algebraic graph theory. These graphical models translate algebraic interactions into combinatorial frameworks, allowing the investigation of both algebraic and graph-theoretic properties, including connectivity, diameter, domination number, clique structure, and spectral parameters. Among such constructions, the commuting graph has emerged as one of the most insightful, as it captures the internal commuting behavior of elements in a visual and measurable form. Initially introduced for groups and subsequently extended to rings and semigroups, commuting graphs have become central objects of study in algebraic graph theory, thereby providing a unifying interface between noncommutative algebra and combinatorics.

Formally, the commuting graph of a group, first introduced in [25] is defined as the simple undirected graph whose vertex set is $G - Z(G)$ where $Z(G)$ denotes the center of the group, and in which two distinct vertices x and y are adjacent if and only if $xy = yx$. The underlying idea is to represent the deviation from commutativity by adopting graph adjacency. This construction was later generalized to rings and semigroups, providing a purely combinatorial framework for investigating the interaction patterns of noncentral elements. Commuting graphs thus serve not only as visualization tools but also as analytic instruments, enabling the examination of several key parameters such as diameter, clique number, chromatic number, and connectedness.

The systematic study of commuting graphs of semigroups is relatively recent compared to its group-theoretic counterparts. In 2011, Araújo, Kinyon, and Konieczny [2] initiated investigations into the commuting graphs of semigroups, focusing on their essential properties, such as the diameter. Their work demonstrated that the commuting graph can reflect intricate algebraic features and offer insight into the semigroup's internal dynamics. Later, Bauer and Greenfeld [6] extended the study to finitely generated semigroups, establishing direct relationships between algebraic parameters (such as the number of generators) and combinatorial invariants (including clique size and graph diameter). These findings underscored the potential of graph-theoretic approaches to reveal new algebraic phenomena.

A complementary direction of research concerns the realizability problem, which investigates which simple graphs can occur as commuting graphs of groups or semigroups. Giudici and Kuzma [13], among others, showed that under appropriate algebraic constraints, a wide variety of graphs can indeed be realized as commuting graphs, thereby highlighting a profound interplay between algebraic structure and graphical representation. This line of inquiry connects semigroup theory with classical areas of combinatorics and has stimulated further research into the representability of abstract graphs by algebraic means.

In recent developments, attention has also shifted to the spectral and topological aspects of commuting graphs. Cheng, Dehmer, Emmert-Streib, Li, and Liu [9] analyzed the spectral properties of commuting graphs over semidihedral groups, computing parameters such as Laplacian energy, eigenvalue distribution, and connectivity indices. Other authors have explored planarity, chromatic number, independence number, and graph energy, thereby extending the study of commuting graphs beyond their structural and combinatorial dimensions into the realm of spectral graph theory. Closely related constructions, such as inclusion-ideal graphs and zero-divisor graphs, have also been introduced to describe ideal-theoretic or annihilator relations in rings and semigroups. Baloda and Kumar [4], for instance, examined inclusion ideal graphs and established meaningful correlations between algebraic properties of semigroups and graph-theoretic parameters such as diameter and chromatic number, demonstrating once again how graphical perspectives enrich algebraic analysis. In [2], Araújo et al. analyzed properties such as diameter, minimal paths, and left paths, demonstrating that graph-theoretic tools can reveal intricate algebraic relationships within semigroups. Following their work, commuting graphs of specific classes of semigroups and various combinatorial or spectral properties of these graphs have been studied extensively in [1, 6, 19, 20, 23, 28].

Within semigroup theory, completely simple semigroups occupy a distinguished position due to their structural representation via the Rees Theorem. The commuting graphs of such semigroups, when realized as Rees matrix semigroups, have been analyzed by several authors in recent years. Paulista and collaborators, among others, have studied numerical and combinatorial parameters, including girth, clique number, chromatic number, and connectivity, thereby providing deeper insight into commutativity in completely simple semigroups and demonstrating their internal symmetry and regularity properties.

In this paper, we introduce the quasi-center based on the definition of the center of a semigroup. The quasi-center offers a more general perspective on commuting behavior, allowing the examination of elements that "almost commute" in a precise algebraic sense. This generalization broadens the scope of classical commuting graph theory and provides a richer framework for capturing additional structural and combinatorial relations. Moreover, we extend the classical theory of commuting graphs to the quasicommuting and extended quasicommuting settings. Specifically, we investigate the quasicommuting and extended quasicommuting graphs of completely simple semigroups and explore their structural and combinatorial properties in comparison with the traditional commuting graphs. The principal objective is to demonstrate that these generalized constructions preserve the essential structural features of classical commuting graphs while simultaneously unveiling new interrelations arising from quasi-commutativity. Through this generalization, we aim to provide a more comprehensive understanding of commutativity in semigroups and lead to further research for the graphical analysis of algebraic structures that deviate from commutativity.

2. Preliminaries

To make this paper self-contained, we want to give some preliminaries on semigroups and graphs. In this section, by S we mean a semigroup.

2.1. Preliminaries on Semigroups

Definition 2.1. [23] The set

$$Z(S) = \{x \in S : xy = yx, \forall y \in S\}$$

is called the center of S.

Definition 2.2. [12] Let $\emptyset \neq I \subseteq S$. If $SI \subseteq I$, then I is called a left ideal of S; if $IS \subseteq I$, then I is called a right ideal of S. If I is both a right ideal and a left ideal of S, then I is called a two-sided ideal of S.

Definition 2.3. [15] Let I be an ideal of S. If $I \subsetneq S$, then I is called a proper ideal of S.

Definition 2.4. [21] Let $x \in S$. If for every $y \in S$ there exists $r \in \mathbb{Z}^+$ such that $xy = y^r x$, then x is called quasicommutative.

Definition 2.5. [21] If for all $x, y \in S$ there exists $r \in \mathbb{Z}^+$ such that $xy = y^r x$, then S is called a quasicommutative semigroup.

Remark 2.1. *Every commutative semigroup is a quasicommutative semigroup. In* [26], *Sorouhesh and Doostie provided an example of a non-commutative quasicommutative semigroup which is not a group of order 9.*

Definition 2.6. [15] Let $e \in S$. If $e^2 = e$, then e is called an idempotent element of S.

Definition 2.7. [16] Suppose S has no proper ideals. If S contains a minimal idempotent with respect to the natural partial order, defined by $e \leq f \Leftrightarrow ef = fe = e$, on the set of all idempotents, then S is called a completely simple semigroup.

Theorem 2.1. [15, Theorem 3.3.1.]*[Rees Theorem]* Let G be a group; I and Λ be two non-empty sets; P be a $\Lambda \times I$ matrix with entries from the group G. Then, $S = (I \times G \times \Lambda)$ is a completely simple semigroup with the following operation:

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$$

Conversely, every completely simple semigroup is isomorphic to a semigroup constructed in this way.

Definition 2.8. [15] A completely simple semigroup S over a group G, together with index sets I and Λ, and a matrix P, equipped with the operation given in Theorem 2.1, is called a Rees matrix semigroup, and is usually denoted by $\mathcal{M}[G; I, \Lambda; P]$.

2.2. Preliminaries on Graphs

Definition 2.9. [14, 11] A graph $G = (V(G), E(G))$ consists of two sets $V(G)$ and $E(G)$, where the elements of $V(G)$ are called vertices. If $E(G)$ is a set of subsets of the form $\{u, v\}$ with $u \neq v \in V(G)$, then G is called an undirected graph. If $E(G) \subseteq V(G) \times V(G)$, then the elements of $E(G)$ are ordered pairs called directed edges (or arcs), and G is called a directed graph (or a digraph).

In this subsection only, instead of $G = (V(G), E(G))$ we shall only write G which will mean a graph, and with D we shall mean a directed graph.

Definition 2.10. [14] If G has neither loops nor multi-edges, then it is called a simple graph. If D has neither loops nor multi-arcs, then it is called a simple digraph.

Definition 2.11. [11] G is called complete if there exists an edge between every pair of its distinct vertices.

Definition 2.12. [5] D is called complete if for every pair of its distinct vertices there exist two arcs of opposite directions.

Definition 2.13. [5] If $(y, x) \in E(D)$, for all $(x, y) \in E(D)$, then D is called a symmetric digraph.

Remark 2.2. *Every symmetric digraph can be treated as an undirected graph. Likewise, every undirected graph can be handled as a symmetric digraph* [5], *as illustrated in Figure 1.*

FIGURE 1. A symmetric digraph is also represented as an undirected graph.

Definition 2.14. [11] If there exists a path between x and y, for every $x, y \in V(G)$, then G is called a connected graph.

Definition 2.15. [14] If the underlying undirected graph of D is connected, then D is called a (weakly) connected digraph.

Definition 2.16. [14] If there exists a directed path between x and y, for every $x, y \in V(D)$, then D is called a strongly connected digraph.

In accordance with Remark 2.2, throughout this paper every undirected edge $\{u, v\}$ is treated as two symmetric arcs, namely (u, v) and (v, u). Hence, every undirected graph is regarded as a symmetric digraph.

Definition 2.17. [23] Let $G = (V, E)$ and $H = (V', E')$ be two graphs. The join of graphs G and H is also another graph represented by $G \triangledown H$, with $V(G \triangledown H) = V \cup V'$ and $E(G \triangledown H) = E \cup E' \cup \{(u, v) : u \in V, v \in V'\}$ as the set of vertices and set of edges, respectively.

Definition 2.18. [14] The vertex coloring of G is a function from $V(G)$ to a set C, whose elements are called colors.

The following three definitions are crucial since they shed light on the properties examined in Section 3 for quasicommuting graphs of completely simple semigroups.

Definition 2.19. [23] Let $G = (V, E)$ be a graph and $H \subseteq V$. If (x, y) is an edge of G, for any distinct $x, y \in H$ then H is called a clique in G. The clique number of G, denoted by $\omega(G)$, is the size of the largest clique in G.

Definition 2.20. [23] Assume that G contains a cycle. The girth of G, denoted by *Girth*(G), is the length of the shortest cycle of G.

Definition 2.21. [23] The minimum number of colors required to color the vertices of G in a way such that adjacent vertices have different colors is called the chromatic number of G, and is denoted by $\chi(G)$.

2.3. Preliminaries on Commuting Graphs of Semigroups

Definition 2.22. [23] Let S be a non-commutative semigroup. Then, the simple undirected graph given by $\triangle(S) = (V(\triangle(S)), E(\triangle(S)))$ where $V(\triangle(S)) = S - Z(S)$ and $E(\triangle(S)) = \{(x, y) : xy = yx,\ x \neq y\}$ is called the commuting graph of S.

Definition 2.23. [23] Let S be a semigroup. Then, the simple undirected graph given by $\triangle^*(S) = (V(\triangle^*(S)), E(\triangle^*(S)))$ where $V(\triangle^*(S)) = S$ and $E(\triangle^*(S)) = \{(x, y) : xy = yx,\ x \neq y\}$ is called the extended commuting graph of S.

The following definition corresponds to a concept created specifically for commuting graphs of semigroups

Definition 2.24. [23] Let S be a semigroup and $L = x_1, x_2, \ldots, x_n$ be a path in $\triangle(S)$. Then, L is called a left path in $\triangle(S)$ if $x_1 \neq x_n$ and $x_1x_i = x_nx_i$ hold for every $i \in \{1, 2, \ldots, n\}$.

3. Main Results

In this section, the results extend those of [23] on commuting graphs of completely simple semigroups to the quasicommuting graphs of completely simple semigroups.

In what follows, we present new definitions and present some results derived from these definitions. For the remainder of this paper, S means a finite semigroup.

3.1. Quasi-center and Quasicommuting Graphs of Semigroups

Definition 3.1. The set

$$Q(S) = \{a \in S : \exists r \in \mathbb{Z}^+, \text{ such that } ab = b^r a, \forall b \in S\}$$

is called the quasi-center of the semigroup S.

Remark 3.1. *Evidently,* $Z(S) \subseteq Q(S)$.

Definition 3.2. Let S be a non-quasicommutative semigroup. Then, the simple digraph given by $\triangle_Q(S) = (V(\triangle_Q(S)), E(\triangle_Q(S)))$ where $V(\triangle_Q(S)) = S - Q(S)$ and $E(\triangle_Q(S)) = \{(x, y) : \exists r \in \mathbb{Z}^+,\ xy = y^r x,\ x \neq y\}$ is called the quasicommuting graph of S.

Definition 3.3. The simple digraph $\triangle_Q^*(S)$, which has the underlying set of S as the vertex set and the set $E(\triangle_Q^*(S)) = \{(x, y) : \exists r \in \mathbb{Z}^+,\ xy = y^r x,\ x \neq y\}$ as the edge set, is called the extended quasicommuting graph of S.

From this point on, for the sake of simplicity, we shall write $E(Q)$ instead of $E(\triangle_Q(S))$ and $E^*(Q)$ instead of $E(\triangle_Q^*(S))$.

Example 3.1. *Let the Cayley table of the semigroup* $S = \{0, a, b, c, d, e\}$ *be as in Table 1.*

Table 1. Cayley Table of S.

$\cdot$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	a
d	0	0	0	0	a	b
e	0	0	a	0	c	0

Since $ce = a \neq e^r c$, *for all* $r \in \mathbb{Z}^+$, S *is not quasicommutative and* $Q(S) = \{0, a, b\}$. *This means the vertex set of* $\triangle_Q(S)$ *is* $S - Q(S) = \{c, d, e\}$. *Since* $cd = 0 = dc$ *and* $ec = 0 = c^2e$, *we get* $E(Q) = \{(c, d), (d, c), (e, c)\}$. *Hence, the quasicommuting graph of* S *is depicted in Figure 2.*

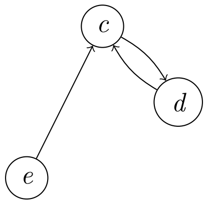

Figure 2. Quasicommuting graph of S.

Moreover, the extended quasicommuting graph of S is as in Figure 3.

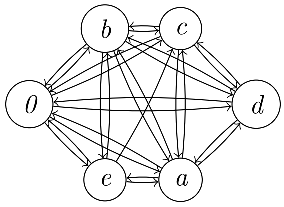

Figure 3. Extended quasicommuting graph of S.

Clearly, $\triangle_Q(S)$ is a subgraph of $\triangle_Q^*(S)$. The following proposition characterizes the relationship between the extended commuting graph and the extended quasicommuting graph of S.

Proposition 3.1. *The extended commuting graph of a semigroup is a subgraph of its extended quasicommuting graph.*

Proof. As noted in Remark 2.2, consider $\triangle^*(S)$ as a symmetric digraph. Let (x, y) be an edge of $\triangle^*(S)$. Hence, $xy = yx$. Therefore, both (x, y) and (y, x) are edges of $\triangle_Q^*(S)$. □

Lemma 3.1. *Let $K_n = (V(K_n), E(K_n))$ be the directed complete graph with n vertices, where $V(K_n) = \{1, 2, \ldots, n\}$. Then, the following are satisfied.*

(1) *If S is quasicommutative, then $\triangle_Q^*(S) \cong K_{|S|}$.*
(2) *If S is not quasicommutative, then $\triangle_Q^*(S) \cong K_{|Q(S)|} \nabla \triangle_Q(S)$.*

Proof. (1) If S is quasicommutative, for all $a, b \in S$, there exists $r \in \mathbb{Z}^+$ such that $ab = b^r a$. Thus, there exists a directed edge between arbitrary pair of vertices. Hence, $\triangle_Q^*(S)$ is a complete digraph.

(2) Let $|S| = m$ and $|Q(S)| = n$. Hence, $K_{|Q(S)|} = K_n$ has n vertices, $\triangle_Q(S)$ has $m - n$ vertices, and $\triangle_Q^*(S)$ has m vertices. Thus, the graph $K_{|Q(S)|} \nabla \triangle_Q(S)$ contains $|V(K_n) \cup (S - Q(S))| = m$ vertices. Consider the sets $S = \{a_1, a_2, \ldots, a_m\}$ and $S - Q(S) = \{a_1, a_2, \ldots, a_{m-n}\}$. Define a function f such that $f(a_i) = a_i$, for all $a_i \in S - Q(S)$, and $f(a_{m-n+k}) = k$, for all $a_{m-n+k} \in Q(S)$. It is straightforward to see that f is a one-to-one and onto function between the sets of vertices of $\triangle_Q^*(S)$ and $K_{|Q(S)|} \nabla \triangle_Q(S)$. Now, observe that there is a one-to-one correspondence between the edges, too.

i) Let $(a_i, a_j) \in (S - Q(S)) \times (S - Q(S))$.
Let $(a_i, a_j) \in E^*(Q)$, for $a_i, a_j \in S - Q(S)$. Then, there exists $r \in \mathbb{Z}^+$ such that $a_i a_j = a_j^r a_i$. Thus, $(a_i, a_j) = (f(a_i), f(a_j)) \in E(Q)$. By the definition of a graph join, the pair $(f(a_i), f(a_j))$ is an edge of $K_{|Q(S)|} \nabla \triangle_Q(S)$.

ii) Let $(a_i, a_j) \in Q(S) \times Q(S)$.
Since $a_i, a_j \in Q(S)$, $(a_i, a_j) \in E^*(Q)$. Furthermore, there exist positive integers t, k where $1 \leq t, k \leq n$ such that $a_i = a_{m-n+k}$ and $a_j = a_{m-n+t}$. Since $f(a_i) = k \in V(K_n)$, $f(a_j) = t \in V(K_n)$, and $K_{|Q(S)|}$ is complete, then $(f(a_i), f(a_j))$ is an edge of K_n. By the definition of graph join, the pair $(f(a_i), f(a_j))$ is an edge of $K_{|Q(S)|} \nabla \triangle_Q(S)$.

iii) Let $(a_i, a_j) \in (S - Q(S)) \times Q(S)$. Since $a_j \in Q(S)$, there exists a positive integer t with $1 \leq t \leq n$ such that $f(a_j) = t$. Since $t \in V(K_n)$ and $f(a_i) = a_i \in S - Q(S)$, $(f(a_i), f(a_j)) = (a_i, t)$ is an edge of $K_{|Q(S)|} \nabla \triangle_Q(S)$ according to the graph join definition.

iv) For the case $(a_i, a_j) \in Q(S) \times (S - Q(S))$, the proof can be examined similarly.

□

3.2. Quasicommuting Graphs of Completely Simple Semigroups

In this subsection, we shall study some properties of quasicommuting graphs and extended quasicommuting graphs of completely simple semigroups.

Proposition 3.2. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup. If $|I| > 1$ or $|\Lambda| > 1$, then S is not quasicommutative and $Q(S) = \emptyset$.*

Proof. In [23], it was observed that S is not commutative for the cases $|I| > 1$ or $|\Lambda| > 1$.

Without loss of generality, assume that $|I| > 1$ and let $i \in I$, $\lambda \in \Lambda$, and $x \in G$. Then, there exists at least one $j \in I$ such that $j \neq i$, and

$$(i, x, \lambda)(j, x, \lambda) = (i, xp_{\lambda j}x, \lambda) \neq (j, (xp_{\lambda j})^r x, \lambda) = (j, x, \lambda)^r (i, x, \lambda)$$

This implies that $Q(S) = \emptyset$. □

Proposition 3.3. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup and $|I| > 1$ or $|\Lambda| > 1$. Consequently, $\triangle(S) \subseteq \triangle_Q(S)$.*

Proof. Let $|I| > 1$. By Proposition 3.2, we know that $S = \mathcal{M}[G; I, \Lambda; P]$ is not quasicommutative. Moreover, $Q(S) = \emptyset$. Since $Z(S) \subseteq Q(S)$, we get $S - Z(S) = S = S - Q(S)$. Thus, $\triangle(S) = \triangle^*(S)$ and $\triangle_Q(S) = \triangle^*_Q(S)$ lead to $\triangle(S) \subseteq \triangle_Q(S)$ by Proposition 3.1. □

Proposition 3.4. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup, $i, j \in I$, $\lambda, \mu \in \Lambda$, and $x, y \in G$, $r \in \mathbb{Z}^+$. Then, $(i, x, \lambda)(j, y, \mu) = (j, y, \mu)^r (i, x, \lambda)$ if and only if $i = j$, $\lambda = \mu$, and $xp_{\lambda i}y = (yp_{\lambda i})^r x$.*

Proof. The case where $r = 1$ is proven in [23]. In what follows, we choose $r > 1$.

$$\begin{aligned}(i, x, \lambda)(j, y, \mu) = (j, y, \mu)^r (i, x, \lambda) &\Leftrightarrow (i, xp_{\lambda j}y, \mu) = (j, (yp_{\mu j})^{r-1} yp_{\mu i}x, \lambda) \\ &\Leftrightarrow i = j, \lambda = \mu, xp_{\lambda j}y = (yp_{\mu j})^{r-1} yp_{\mu i}x \\ &\Leftrightarrow i = j, \lambda = \mu, xp_{\lambda i}y = (yp_{\lambda i})^{r-1} yp_{\lambda i}x \\ &\Leftrightarrow i = j, \lambda = \mu, xp_{\lambda i}y = (yp_{\lambda i})^r x\end{aligned}$$

□

Proposition 3.5. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup, $i \in I$, $\lambda \in \Lambda$, $x, y \in G$, and $r \in \mathbb{Z}^+$. Then, $xy = y^r x$ if and only if $(i, p_{\lambda i}^{-1} x, \lambda)(i, p_{\lambda i}^{-1} y, \lambda) = (i, p_{\lambda i}^{-1} y, \lambda)^r (i, p_{\lambda i}^{-1} x, \lambda)$.*

Proof. The situation $r = 1$ is also investigated in [23]. Assume that $r > 1$.

$$\begin{aligned} xy = y^r x &\Leftrightarrow p_{\lambda i}^{-1} xy = p_{\lambda i}^{-1} y^r x = p_{\lambda i}^{-1} \underbrace{y \dots y}_{r\text{-times}} x \\ &\Leftrightarrow p_{\lambda i}^{-1} x p_{\lambda i} p_{\lambda i}^{-1} y = \underbrace{\underbrace{p_{\lambda i}^{-1} y p_{\lambda i}} \underbrace{p_{\lambda i}^{-1} y p_{\lambda i}} \dots \underbrace{p_{\lambda i}^{-1} y p_{\lambda i}}}_{r\text{-times}} p_{\lambda i}^{-1} x \\ &\Leftrightarrow p_{\lambda i}^{-1} x p_{\lambda i} p_{\lambda i}^{-1} y = (p_{\lambda i}^{-1} y p_{\lambda i})^r p_{\lambda i}^{-1} x \\ &\Leftrightarrow (i, p_{\lambda i}^{-1} x p_{\lambda i} p_{\lambda i}^{-1} y, \lambda) = (i, (p_{\lambda i}^{-1} y p_{\lambda i})^r p_{\lambda i}^{-1} x, \lambda) \\ &\Leftrightarrow (i, p_{\lambda i}^{-1} x, \lambda)(i, p_{\lambda i}^{-1} y, \lambda) = (i, p_{\lambda i}^{-1} y, \lambda)^r (i, p_{\lambda i}^{-1} x, \lambda) \end{aligned}$$

□

For the next Theorem, $C_{i\lambda}$ will denote the set $C_{i\lambda} = \{(i, x, \lambda) : x \in G\} \subseteq \mathcal{M}[G; I, \Lambda; P]$. It is pretty straightforward to see that $C_{i\lambda}$ is a group with $(i, p_{\lambda i}^{-1}, \lambda)$ as its identity.

Theorem 3.1. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup, and $|I| > 1$ or $|\Lambda| > 1$. Then, $\triangle_Q(S)$ is not connected, and its connected components are subgraphs induced by the set $C_{i\lambda}$, expressed by $\triangle_Q(C_{i\lambda})$. Moreover, all connected components of $\triangle_Q(S)$ are isomorphic to $\triangle_Q^*(G)$, and*

$$Diam(\triangle_Q(C_{i\lambda})) = \begin{cases} 0, & G \text{ is the trivial group} \\ 1, & G \text{ is a non-trivial quasicommutative group} \\ 2, & G \text{ is a non-quasicommutative group} \end{cases}$$

Proof. Let $x \in G \backslash \{1_G\}$. Since $p_{\lambda i} x 1_G = 1_G^r p_{\lambda i} x = 1_G(p_{\lambda i} x)$, for any $r \in \mathbb{Z}^+$, by Proposition 3.5, we conclude that

$$\begin{aligned} (i, p_{\lambda i}^{-1}(p_{\lambda i})x, \lambda)(i, p_{\lambda i}^{-1} 1_G, \lambda) &= (i, x, \lambda)(i, p_{\lambda i}^{-1}, \lambda) \\ &= (i, p_{\lambda i}^{-1}, \lambda)^r (i, x, \lambda) \\ &= (i, p_{\lambda i}^{-1} 1_G, \lambda)^r (i, p_{\lambda i}^{-1}(p_{\lambda i})x, \lambda) \end{aligned}$$

This means $((i, x, \lambda), (i, p_{\lambda i}^{-1}, \lambda))$ is a directed edge of $\triangle_Q(S)$. Similarly, since the equality $1_G(p_{\lambda i} x) = (p_{\lambda i} x)1_G$, via Proposition 3.5,

$$(i, p_{\lambda i}^{-1} 1_G, \lambda)(i, p_{\lambda i}^{-1}(p_{\lambda i} x), \lambda) = (i, p_{\lambda i}^{-1}(p_{\lambda i} x), \lambda)(i, p_{\lambda i}^{-1} 1_G, \lambda)$$

Hence, $((i, p_{\lambda i}^{-1}, \lambda), (i, x, \lambda))$ is also a directed edge of $\triangle_Q(S)$. Since all $(i, x, \lambda) \in C_{i\lambda}$ are adjacent to $(i, p_{\lambda i}^{-1}, \lambda)$, they all belong to the same connected component of $\triangle_Q(S)$. We aim to show that every vertex of any path with $(i, p_{\lambda i}^{-1}, \lambda)$ as its initial vertex, also belongs to the set $C_{i\lambda}$. Let

$$(i, p_{\lambda i}^{-1}, \lambda) = (i_1, x_1, \lambda_1), (i_2, x_2, \lambda_2), \dots, (i_n, x_n, \lambda_n)$$

be a path with $(i, p_{\lambda i}^{-1}, \lambda)$ as its initial vertex. Then, there exists a positive integer r such that

$$(i_j, x_j, \lambda_j)(i_{j+1}, x_{j+1}, \lambda_{j+1}) = (i_{j+1}, x_{j+1}, \lambda_{j+1})^r (i_j, x_j, \lambda_j)$$

for all $j \in \{1, \dots, n-1\}$. By Proposition 3.4, $i = i_j$, $\lambda = \lambda_j$, for all $j \in \{1, \dots, n\}$. Therefore, we conclude that each vertex of this directed path also belongs to the set $C_{i\lambda}$. Hence, the subgraph induced by $C_{i\lambda}$ is strongly connected. We next show that $\triangle_Q(C_{i\lambda})$ is isomorphic to $\triangle_Q^*(G)$. It is well established that ϕ is a group isomorphism from G to $C_{i\lambda}$, where the operation is defined by $\phi(x) = (i, p_{\lambda i}^{-1}x, \lambda)$, for $x \in G$. (x, y) is an edge of $\triangle_Q^*(G)$, then there exists $r \in \mathbb{Z}^+$ such that $xy = y^r x$. By Proposition 3.5,

$$\begin{aligned}\phi(x)\phi(y) &= (i, p_{\lambda i}^{-1}x, \lambda)(i, p_{\lambda i}^{-1}y, \lambda)\\ &= (i, p_{\lambda i}^{-1}y, \lambda)^r(i, p_{\lambda i}^{-1}x, \lambda)\\ &= \phi(y)^r\phi(x)\end{aligned}$$

is satisfied. Therefore, $(\phi(x), \phi(y))$ is a directed edge of $\triangle_Q(C_{i\lambda})$. This concludes that ϕ is a graph isomorphism. It remains to calculate $Diam(\triangle_Q(C_{i\lambda}))$. Since $\triangle_Q(C_{i\lambda}) \cong \triangle_Q^*(G)$, they have the same diameter. If $G = \{1_G\}$, then $\triangle_Q^*(G)$ has only one vertex. Therefore, its diameter is zero. If G is a non-trivial quasicommuting group, then it has more than one vertex that are all adjacent to each other. This implies that its diameter is equal to 1. For the final case, assume that G is a non-quasicommutative group. Since $1_G x = x 1_G$, for all $x \in G$, $(x, 1_G)$ and $(1_G, x)$ are edges of $\triangle_Q^*(G)$. We deduce that the maximum distance between any two vertices is at most two. Since G is not quasicommutative, there exist $x, y \in G$ such that $xy \neq y^r x$, for all $r \in \mathbb{Z}^+$. This concludes that $Diam(\triangle_Q^*(G)) = Diam(\triangle_Q(C_{i\lambda})) = 2$. □

Corollary 3.1. *The quasicommuting graph of $S = \mathcal{M}[G; I, \Lambda; P]$ has $|I| \cdot |\Lambda|$ connected components.*

Theorem 3.2. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup and $|I| > 1$ or $|\Lambda| > 1$. If G is quasicommutative, then $\omega(\triangle_Q(S)) = |G|$; otherwise, $\omega(\triangle_Q(S)) = |Q(G)| + \omega(\triangle_Q(S))$.*

Proof. Clique number of a graph which is not connected, is equal to the maximum clique number among its connected components. Furthermore, we have demonstrated in Theorem 3.1, all connected components of quasicommuting graph of $S = \mathcal{M}[G; I, \Lambda; P]$ are isomorphic to $\triangle_Q^*(G)$. To prove our claim, we analyze two cases.

For the first case, suppose that G is quasicommutative. Then, $\triangle_Q^*(G) \cong K_{|G|}$. The clique number of a complete graph is equal to the number of its vertices. Thus, $\omega(\triangle_Q(S)) = |G|$.

For the second case, assume that G is not a quasicommutative group. It follows from Lemma 3.1, $\triangle_Q^*(G) \cong K_{|Q(G)|} \triangledown \triangle_Q(G)$. The clique number of the join of two graphs is equal to the sum of the clique numbers of those two graphs. Hence, $\omega(\triangle_Q^*(G)) = \omega(K_{|Q(G)|}) + \omega(\triangle_Q(G)) = |Q(G)| + \omega(\triangle_Q(G))$. □

Theorem 3.3. *Let G be a group. Then, $\triangle_Q^*(G)$ contains a cycle if and only if $|G| \geq 2$ and $Girth(G) = 2$.*

Proof. There are two cases to be considered in the proof of this theorem. For the first case, consider the quasicommutative group G. If $|G| = 1$, then $\triangle_Q^*(G)$ only has one vertex. Thus, it cannot contain a cycle. Assume that $|G| \geq 2$. By Lemma 3.1, we know that $\triangle_Q^*(G)$ is isomorphic to $K_{|G|}$. Then, since $\triangle_Q^*(G)$ is a complete graph, it contains at least one cycle of length 2. We conclude that $Girth(G) = 2$. Consider the second case, suppose that G is not quasicommutative. This implies that G consists of at least three elements. Suppose $x \in G$ such that $x \neq 1_G$. Since $x1_G = 1_G x$, $x, 1_G, x$ is a directed cycle of length 2 in $\triangle_Q^*(G)$. It follows that $Girth(G) = 2$. □

The following Corollary is a direct result of Theorem 3.3.

Corollary 3.2. *Let $S = \mathcal{M}[G; I, \Lambda; P]$. If $|G| = 1$, then $\triangle_Q(\mathcal{M}[G; I, \Lambda; P])$ does not contain a cycle. If $|G| \geq 2$, then $\triangle_Q(\mathcal{M}[G; I, \Lambda; P])$ contains a cycle of length 2.*

Theorem 3.4. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup and $|I| > 1$ or $|\Lambda| > 1$. If G is quasicommutative, then $\chi(\triangle_Q(S)) = |G|$; otherwise, $\chi(\triangle_Q(S)) = |Q(G)| + \chi(\triangle_Q(G))$.*

Proof. The chromatic number of a disconnected graph is determined by the maximum chromatic number among its connected components. Additionally, in Theorem 3.1, we established that all connected components of the quasicommuting graph of $S = \mathcal{M}[G; I, \Lambda; P]$ are isomorphic to $\triangle_Q^*(G)$. Two cases are considered.

For the first case, we assume that G is quasicommutative. Then, we have $\triangle_Q^*(G) \cong K_{|G|}$. Therefore, $\chi(\triangle_Q(S)) = |G|$.

For the second case, consider G to be a non-quasicommutative group. According to Lemma 3.1, $\triangle_Q^*(G) \cong K_{|Q(G)|} \nabla \triangle_Q(G)$. The chromatic number of the join of two graphs is equal to the sum of their respective chromatic numbers. Consequently, $\chi(\triangle_Q(S)) = |Q(G)| + \chi(\triangle_Q(G))$. □

Theorem 3.5. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be non-quasicommutative. Then, $\triangle_Q(\mathcal{M}[G; I, \Lambda; P])$ does not contain a left path.*

Proof. Assume that

$$(i_1, x_1, \lambda_1), (i_2, x_2, \lambda_2), \ldots, (i_m, x_m, \lambda_m)$$

is a left path of $\triangle_Q(S) = \triangle_Q(\mathcal{M}[G; I, \Lambda; P])$. Hence, for all $k \in \{1, 2, \ldots, m-1\}$

$$(i_k, x_k, \lambda_k)(i_{k+1}, x_{k+1}, \lambda_{k+1}) = (i_{k+1}, x_{k+1}, \lambda_{k+1})^r (i_k, x_k, \lambda_k)$$

for some $r \in \mathbb{Z}^+$. Then by Proposition 3.4, we know that $i_k = i_{k+1}$ and $\lambda_k = \lambda_{k+1}$, for all $k \in \{1, 2, \ldots, m-1\}$. This impplies $i_1 = i_2 = \cdots = i_m$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_m$. For ease of exposition, denote $i = i_k$ and $\lambda = \lambda_k$, for all $k \in \{1, 2, \ldots, m\}$. Hence,

$$\begin{aligned}
&(i_1, x_1, \lambda_1)(i_1, x_1, \lambda_1) = (i_m, x_m, \lambda_m)(i_1, x_1, \lambda_1) \\
&\Rightarrow (i, x_1, \lambda)(i, x_1, \lambda) = (i, x_m, \lambda)(i, x_1, \lambda) \\
&\Rightarrow (i, x_1 p_{\lambda i} x_1, \lambda) = (i, x_m p_{\lambda i} x_1, \lambda) \\
&\Rightarrow x_1 p_{\lambda i} x_1 = x_m p_{\lambda i} x_1 \\
&\Rightarrow x_1 = x_m
\end{aligned}$$

which contradicts the left path definition. Thus, $\triangle_Q(\mathcal{M}[G; I, \Lambda; P])$ can not contain a left path. □

Corollary 3.3. *The quasicommuting graph of a finite non-quasicommutative group does not contain a left path.*

Proof. In Theorem 3.5, we have demonstrated that $\triangle_Q(\mathcal{M}[G; I, \Lambda; P])$ does not contain a left path, for a non-quasicommutative $\mathcal{M}[G; I, \Lambda; P]$. This is also true when $|I| = |\Lambda| = 1$, which further implies $G \cong \mathcal{M}[G; I, \Lambda; P]$. Thus, $\triangle_Q(G)$ does not contain a left path. □

Corollary 3.4. *For every $n \in \mathbb{Z}^+$, there exists a completely simple semigroup S such that $\omega(\triangle_Q(S)) = n$.*

Proof. Let $n \in \mathbb{Z}^+$ and C_n be the cyclic group of order n. Then, C_n is quasicommutative. On the other hand, choose $|I| > 1$ or $|\Lambda| > 1$. Then, $\mathcal{M}[C_n; I, \Lambda; P]$ is a completely simple semigroup that is not quasicommutative via Proposition 3.2. By Theorem 3.2, $\omega(\mathcal{M}[C_n; I, \Lambda; P]) = |C_n| = n$. □

Corollary 3.5. *For every $n \in \mathbb{Z}^+$, there exists a completely simple semigroup S such that $\chi(\triangle_Q(S)) = n$.*

Proof. Consider the case where $|I| > 1$ or $|\Lambda| > 1$ with C_n being the cyclic group of order n. Consequently, $\mathcal{M}[C_n; I, \Lambda; P]$ is a completely simple semigroup that is not quasicommutative. Hence, by Theorem 3.4,
$\chi(\triangle_Q(\mathcal{M}[C_n; I, \Lambda; P])) = |C_n| = n$. □

Theorem 3.6. *Let $\triangle$ be a simple graph. Then, $\triangle$ is the quasicommuting graph of a completely simple semigroup if and only if one of the following conditions is satisfied.*

(1) *$\triangle$ is a quasicommuting graph of a group.*

(2) *$\triangle$ has more than one connected component, which are all isomorphic to the extended quasicommuting graph of a group.*

Proof. We start the proof by proving the "only if" part. Assume that $\triangle = \triangle_Q(S)$ is a quasicommuting graph of a completely simple semigroup S. Then, there exist index sets I and Λ; and a regular matrix P whose entries are elements of the group G, such that $S = \mathcal{M}[G; I, \Lambda; P]$.

First, suppose that $|I| = |\Lambda| = 1$. Thus, $G \cong \mathcal{M}[G; I, \Lambda; P]$. This implies that S is a group, which yields item 1.

Secondly, assume that $|I| > 1$ or $|\Lambda| > 1$. Then, by Theorem 3.1 and Corollary 3.1, we know that $\mathcal{M}[G; I, \Lambda; P]$ has $|I|.|\Lambda|$ connected components, which are all isomorphic to $\triangle_Q^*(G)$. This implies item 2.

The "if" part. Let $\triangle = \triangle_Q(G)$ be a quasicommuting graph of a group. Since every group is a completely simple semigroup, $\triangle$ is a quasicommuting graph of a semigroup.

Suppose that $\triangle$ is a graph which has $m \geq 2$ connected components that are all isomorphic to $\triangle_Q^*(G)$, where G is a group. Take $|I| = m$ and $|\Lambda| = 1$; and P as a $\Lambda \times I$ matrix. By Corollary 3.1, $\triangle_Q(\mathcal{M}[G; I, \Lambda; P])$ has $|I|.|\Lambda| = m$ connected components that are isomorphic to $\triangle_Q^*(G)$. It follows that $\triangle \cong \triangle_Q(\mathcal{M}[G; I, \Lambda; P])$. □

4. **Conclusions**

In this paper, we first formulate the definitions quasi-center of a semigroup and quasicommuting graph of a semigroup, drawing upon its center and commuting graph, respectively. We have established under what conditions a completely simple semigroup is not quasicommutative. We demonstrated that the commuting graph of a completely simple semigroup S is a subgraph of the quasicommuting graph of the completely simple semigroup S. Subsequently, we generalize several known results of commuting graphs of completely simple semigroups to quasicommuting graphs of completely simple semigroups. Furthermore, we have investigated clique numbers, diameters, girths, and chromatic numbers of the quasicommuting graphs of completely simple semigroups. We consider the necessary and sufficient conditions that must be satisfied for a graph to qualify as the quasicommuting graph corresponding to a completely simple semigroup.

Author's Contribitions

This paper is derived from the first author's masters thesis supervised by the second author. They all read and approved the final version of the paper.

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